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Symmetric cubical sets

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ABSTRACT

We introduce a new cubical model for homotopy types. More precisely, we will define a category $\mathcal{Q}_\Sigma$ with the following features: $\mathcal{Q}_\Sigma$ is a PROP containing the classical box category as a subcategory; the category $q_\Sigma\mathbf{Set}$ of presheaves of sets on $\mathcal{Q}_\Sigma$ models the homotopy category; and combinatorial symmetric monoidal model categories with cofibrant unit have homotopically well-behaved $q_\Sigma\mathbf{Set}$ enrichments.

1. Introduction

There are two well-known classical models for the homotopy category: one can start with the category **Top** of (compactly generated weak Hausdorff) topological spaces, associate a CW approximation $\gamma X \to X$ to each space X, and take Ho $\mathbf{Top}(X, Y)$ to be the homotopy classes of maps between γX and γY. The Whitehead theorem implies that weak equivalences between CW complexes are homotopy equivalences, so Ho **Top** is the localization of **Top** at the category of weak homotopy equivalences. Alternatively, one can use the category $s\mathbf{Set}$ of simplicial sets, Kan approximations, and $\Delta[1]$-homotopy—with the modification that Kan approximations are made on the right—to construct a homotopy category of simplicial sets [24]. The geometric realization-singular set adjunction

$$|-| : s\mathbf{Set} \rightleftarrows \mathbf{Top} : \mathrm{Sing}$$

is a Quillen equivalence: it descends to an equivalence of homotopy categories and preserves the homotopy types of mapping spaces [16,31]. Any homotopy-theoretic result true in **Top** is thus true in $s\mathbf{Set}$ and vice versa. In a more modern language, one can view **Top** and $s\mathbf{Set}$ as two presentations of the same $(\infty, 1)$-category, using whichever is more convenient for the application at hand.

One advantage of simplicial sets is that the category $s\mathbf{Set}$ is a presheaf topos, unlike **Top**. (One great disadvantage is that almost no space comes "in nature" as a simplicial set, and many geometric constructions rely on **Top**.) In fact, the category Δ of finite nonempty totally ordered sets is not the only site upon which we may model the homotopy category. For example, the cubical category $\mathcal{Q}$—the category of posets $\{0 < 1\}^n$, $n \geq 0$ with maps those maps given by deleting coordinates or inserting 0s and 1s—also models spaces via the associated category $q\mathbf{Set}$ of presheaves on $\mathcal{Q}$. This result, in the language of model categories, is relatively recent. Denis-Charles Cisinski and Georges Maltsiniotis, building on conjectures of Grothendieck [18], have given a unified perspective of categorical homotopy theory and presheaf models for the homotopy category in [9,27,28] (also see [23])—one side benefit is a straightforward demonstration that $q\mathbf{Set}$ is a model for the homotopy category.

In the cubical category $\mathcal{Q}$, the product of two cubes is again a cube. In $s\mathbf{Set}$, the product of two representable functors (i.e., two simplices) is not representable—the decomposition of $\Delta[n]\times\Delta[m]$ into $n+m$-simplices is a fundamental construction in simplicial theory. This straightforward fact about $\mathcal{Q}$ considerably simplifies the project of finding a spatial enrichment in an arbitrary homotopical category: "n-cubes" of a cubical mapping space are simply n-fold homotopies. Of course, cubical sets come with their own disadvantages: without adding extra degeneracies, the analogous Dold–Kan correspondence fails [6].

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Moreover, the convolution monoidal structure on cubical sets is not symmetric. In order to remedy these there is a menagerie of cubical categories containing $\mathscr{Q}$ as a subcategory [17]. In this paper, we will add one more category $\mathscr{Q}_\Sigma$ to the zoo, with some useful features:

(1) $\mathscr{Q}_\Sigma$ is symmetric monoidal—in fact, it is a PROP in **Set**—and hence the category $q_\Sigma\mathbf{Set} = \mathbf{Set}^{\mathscr{Q}_\Sigma{}^{\mathrm{op}}}$ is symmetric monoidal.
(2) There are left Quillen equivalences $i_! : q\mathbf{Set} \to q_\Sigma\mathbf{Set}$ and $|-|_\Sigma : q_\Sigma\mathbf{Set} \to \mathbf{Top}$. These are strong monoidal and strong symmetric monoidal, respectively (Theorem 4.18).
(3) Any combinatorial symmetric monoidal model category $\mathscr{C}$ with cofibrant unit may be enriched over $q_\Sigma\mathbf{Set}$. In fact, $\mathscr{C}$ can be realized as a symmetric monoidal model category of presheaves of cubical sets over a symmetric monoidal site with the monoidal structure given by Day convolution; this is a refinement of a theorem of Daniel Dugger [11].

We will show (3) in a future paper; it constitutes the second and third chapter of the author's dissertation [22].

The plan of this paper is as follows. In Section 2, we describe the category $\mathscr{Q}_\Sigma$ as a PROP. In order to lift the model structure on $q\mathbf{Set}$ to $q_\Sigma\mathbf{Set}$, we need to do some careful combinatorial analysis of skeleta of symmetric cubical sets. This occupies Section 3 and relies a great deal on the methods of Cisinski [9] and Berger and Moerdijk [2]. In Section 4, we prove that $q_\Sigma\mathbf{Set}$ models the homotopy category by lifting the model structure on cubical sets. With a view towards structure theorems about model categories, we prove a regularity theorem in Section 5. Finally, we review some facts about Day convolution in the Appendix.

2. The symmetric cubical site

2.1. Cubical monoids

There are several cubical categories in the literature, each generated by a selection of face, degeneracy and possibly symmetry maps; Grandis and Mauri study a zoology of cubical sites in [17]. Our site $\mathscr{Q}_\Sigma$, defined below, is a novel addition. The cubical category with the fewest maps has as objects the posets $[n] = \{0 < 1\}^n$, $n \geq 0$; a map $\{0 < 1\}^n \to \{0 < 1\}^m$ may erase coordinates (degeneracies) and insert 0 or 1, but may *not* repeat coordinates or change their order. This is the classical "box category"; we denote it by $\mathscr{Q}$ and write $q\mathbf{Set}$ for the associated category of presheaves of sets on $\mathscr{Q}$. We write $\square^n$ for the representable presheaf $\mathscr{Q}(-, [n])$. The category $\mathscr{Q}$ has a monoidal structure given by concatenation. Viewed as a PRO [3], its algebras in a monoidal category $(\mathscr{C}, \otimes, e)$ are diagrams

$$\begin{tikzcd} e \amalg e \arrow[r] \arrow[rr, bend right, "\mathrm{id} \amalg \mathrm{id}"'] & I \arrow[r] & e. \end{tikzcd} \tag{2.1}$$

This notion is found in [9]. We will call diagrams of the shape (2.1) *intervals.*

In [5,4], Brown and Higgins study a cubical site with an extra degeneracy called a "connection"; the connection maps are generated by the logical conjunction

$$- \wedge - : \{0 < 1\}^2 \to \{0 < 1\} \tag{2.2}$$

with $x \wedge y = 1$ if and only if $x = y = 1$. Connections were introduced earlier by Brown and Spencer in [7] in the context of double groupoids. Imposing the structure of a connection on an interval motivates the following definition:

Definition 2.1. Suppose $(\mathscr{C}, \otimes, e)$ is a monoidal category. A *cubical monoid* in $\mathscr{C}$ is a diagram

$$\begin{tikzcd} e \amalg e \arrow[r, "d_0 \amalg d_1"] \arrow[rr, bend right, "\mathrm{id} \amalg \mathrm{id}"'] & I \arrow[r, "s"] & e \end{tikzcd} \tag{2.3}$$

together with a map $\mu : I \otimes I \to I$ so that

(1) The map μ makes I an associative monoid with unit d_1.
(2) The map s is a monoid map.
(3) The map $d_0 : e \to I$ is absorbing, i.e., the diagram

$$\begin{tikzcd} I \otimes e \arrow[r, "\mathrm{id}_I \otimes d_0"] \arrow[d, "s \otimes \mathrm{id}_e"'] & I \otimes I \arrow[d, "\mu"] & e \otimes I \arrow[l, "d_0 \otimes \mathrm{id}_I"'] \arrow[d, "\mathrm{id}_e \otimes s"] \\ e \arrow[r, "d_0"'] & I & e \arrow[l, "d_0"] \end{tikzcd} \tag{2.4}$$

commutes.

We will sometimes abuse notation and simply say that I is a cubical monoid. A map of cubical monoids $I \to J$ is a map in $\mathscr{C}$ commuting with all the structure data. We write $\mathbf{qMon}(\mathscr{C})$ for the category of cubical monoids in $\mathscr{C}$.

Example 2.2. Suppose $(\mathscr{C}, \otimes, e)$ is a monoidal category.

(1) The unit e is a cubical monoid with $d_0 = d_1 = s = \mathrm{id}_e$ and $\mu : e \otimes e \to e$ given by the coherence isomorphisms of $\mathscr{C}$. This is the terminal cubical monoid in $\mathscr{C}$.
(2) The coproduct $e \amalg e$ is a cubical monoid with d_0 and d_1 given by the inclusion of each summand. The multiplication μ and degeneracy s are forced. This is the initial cubical monoid in $\mathscr{C}$.

(3) The 1-simplex $\Delta[1] \in s\mathbf{Set}$ is a cubical monoid via the connection map (2.2).
(4) If $F : \mathscr{C} \to \mathscr{D}$ is lax symmetric monoidal and I is a cubical monoid, then FI is a cubical monoid; so, for example, the normalized chains on $\Delta[1]$ are a cubical monoid in chain complexes.

There is an alternative description of cubical monoids shown to the author by Reid Barton. First observe that the category $([1], \wedge, 1)$ has the structure of a monoidal category (here $[1] = \{0 < 1\}$). Suppose $(\mathscr{C}, \otimes, e)$ is a monoidal category in which $\mathscr{C}$ has all small colimits and $- \otimes -$ preserves colimits in each variable. We may then equip the category $\mathscr{C}^{[1]}$ of arrows in $\mathscr{C}$ with the Day convolution monoidal structure. If $f : A \to B$ and $g : X \to Y$ are arrows in $\mathscr{C}$, their product $f \odot g$ is the usual pushout-product

$$f \odot g : A \otimes Y \coprod_{A \otimes X} B \otimes X \to B \otimes Y. \tag{2.5}$$

The unit is the unique map $\emptyset \to e$. Note that $\emptyset \to e$ and $\mathrm{id}_e : e \to e$ are both monoids in $\mathscr{C}^{[1]}$.

Proposition 2.3. *The category* $\mathbf{qMon}(\mathscr{C})$ *of cubical monoids in* $\mathscr{C}$ *is equivalent to the category of monoids of the form* $d_0 : e \to I$ *intervening in a diagram*

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & e \arrow[r, "\mathrm{id}_e"] \arrow[d, "d_0"] & e \arrow[d, "\mathrm{id}_e"] \\ e \arrow[r, "d_1"'] & I \arrow[r, "s"'] & e \end{tikzcd} \tag{2.6}$$

of monoids in $\mathscr{C}^{[1]}$.

Note that the condition that $d_0 : e \to I$ be absorbing, in the language of the product $\odot$, becomes the commutativity of the diagram

$$\begin{tikzcd} I \otimes e \coprod_{e \otimes e} e \otimes I \arrow[r, "d_0 \odot d_0"] \arrow[d] & I \otimes I \arrow[d] \\ e \arrow[r, "d_0"'] & I. \end{tikzcd}$$

It is forced by requiring $d_0 : e \to I$ to be a monoid in $\mathscr{C}^{[1]}$.

Our symmetric cubical site $\mathscr{Q}_\Sigma$ parameterizes cubical monoids. We could define it as follows:

Provisional Definition 2.4. The category $\mathscr{Q}_\Sigma$ is the PROP whose category of algebras in a symmetric monoidal category $(\mathscr{C}, \otimes, e)$ is the category $\mathbf{qMon}(\mathscr{C})$.

However, while this definition is conceptually satisfactory, it does not give a description of the maps in $\mathscr{Q}_\Sigma$. We will define $\mathscr{Q}_\Sigma$ in terms of generators and relations in Section 2.2, where Definition 2.4 will be proved as Proposition 2.15. In the meantime, we note an immediate consequence of Definition 2.4: since each cubical monoid yields an interval by forgetting structure, we have a strict monoidal functor $i : \mathscr{Q} \to \mathscr{Q}_\Sigma$.

Remark 2.5. Note that cubical monoids are not abelian, but $\mathscr{Q}_\Sigma$ is to be a symmetric monoidal category. The PRO $\mathscr{Q}'$ whose algebras are cubical monoids in an arbitrary monoidal category is straightforward to describe: it is the cubical site obtained from $\mathscr{Q}$ by adjoining connection maps and the appropriate relations (see [17] or below). Maltsiniotis studies the homotopy theory of $\mathscr{Q}'$-presheaves in [28]. The construction of $\mathscr{Q}_\Sigma$ is analogous to the symmetrization of a non-Σ operad [29]. Note however that $\mathscr{Q}_\Sigma$ is not freely generated by an operad, since it includes a 1–0 operation corresponding to the degeneracy $s : I \to e$. This makes symmetrization more complicated: as we will see below, permutations can be moved past the map s, but not past connections.

2.2. Maps in the category $\mathscr{Q}_\Sigma$

Definition 2.6. Suppose S is a set of symbols not containing 0 or 1. A *formal cubical product* on S is either

(1) an ordered connection of elements of S, none occurring more than once (i.e., a list of symbols in S separated by $\wedge$); or
(2) the numeral 0 or 1.

A *formal cubical* (m, n)*-product* is an n-tuple $(y_1, \ldots, y_n)$ of formal cubical products on $\{x_1, \ldots, x_m\}$ so that no symbol x_i occurs in more than one formal cubical product y_j. Write $\mathscr{Q}_\Sigma(\llbracket m \rrbracket, \llbracket n \rrbracket)$ for the set of all formal cubical (m, n) products. By convention, $\mathscr{Q}_\Sigma(\llbracket m \rrbracket, \llbracket 0 \rrbracket)$ is a single point.

For example, the following are formal cubical (3, 2)-products:

$$(x_1, 0) \qquad (1, x_3 \wedge x_2) \qquad (1, 1) \qquad (x_1 \wedge x_3, x_2).$$

However, (x_1, x_1) is not a formal cubical product as the symbol x_1 occurs more than once.

Definition 2.7. The *identity formal* (n, n)*-product* is the n-tuple $(x_1, \dots, x_n)$. Suppose X and Y are formal cubical (ℓ, m)- and (m, n)-products, respectively. The *composition* $Y \circ X$ is defined as follows:

(1) Replace any occurrence of the symbol x_i in Y with the ith entry of X.
(2) Delete each occurrence of the symbol 1 in each connection of length at least two.
(3) Replace each connection containing 0 with the numeral 0.

For example, we have the following compositions:

$$(x_3, x_1 \wedge x_2) \circ (0, x_1, x_5) = (x_5, 0) \qquad (x_2 \wedge x_1) \circ (x_1 \wedge x_2, x_3) = (x_3 \wedge x_1 \wedge x_2)$$
$$(x_1 \wedge x_2) \circ (1, 1) = (1) \qquad (0, x_1 \wedge x_4) \circ (x_{10}, 0, 0, 1, x_3) = (0, x_{10}).$$

This makes $\mathcal{Q}_\Sigma$ a category with objects $[\![n]\!]$, $n \geq 0$ and maps $[\![m]\!] \to [\![n]\!]$ formal cubical (m, n)-products. We call $\mathcal{Q}_\Sigma$ the *extended cubical category* and presheaves on $\mathcal{Q}_\Sigma$ *extended cubical sets;* we notate the category of extended cubical sets as $q_\Sigma\mathbf{Set}$. We write $\square^n_\Sigma$ for the representable presheaf $\mathcal{Q}_\Sigma(-, [\![n]\!])$. To complete the description of $\mathcal{Q}_\Sigma$ as a PROP we need its symmetric strict monoidal structure:

Definition 2.8. Define $[\![m]\!] \oplus [\![n]\!] = [\![m + n]\!]$. Suppose X_i is a formal cubical (m_i, n_i)-product, $i = 1, 2$. Define $X_1 \oplus X_2$ as follows:

(1) Replace each x_j in X_2 by x_{j+m_1} to form X_2'.
(2) Concatenate X_1 and X_2'.

The symmetry $[\![m]\!] \oplus [\![n]\!] \to [\![n]\!] \oplus [\![m]\!]$ is the formal product

$$(x_{m+1}, x_{m+2}, \dots, x_{m+n}, x_1, x_2, \dots, x_m).$$

For example,

$$\big((x_1 \wedge x_2) : [\![2]\!] \to [\![1]\!]\big) \oplus \big((0, x_1) : [\![1]\!] \to [\![2]\!]\big) = (x_1 \wedge x_2, 0, x_3) : [\![3]\!] \to [\![3]\!].$$

2.3. Generators and relations in $\mathcal{Q}_\Sigma$

In order to describe skeletal filtrations on extended cubical sets, we need a presentation of $\mathcal{Q}_\Sigma$. The relations we list are a subset of those in [17].

Definition 2.9. Suppose $n \geq 0$, $1 \leq i \leq n + 1$, and $\varepsilon = 0, 1$. Define maps $\delta^{i,\varepsilon}_n$ and σ^i_n by the formal products

$$\begin{aligned} \delta^{i,\varepsilon}_n &= (x_1, \dots, x_{i-1}, \varepsilon, x_i, \dots, x_n) : [\![n]\!] \to [\![n+1]\!] \\ \sigma^i_n &= (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+1}) : [\![n+1]\!] \to [\![n]\!]. \end{aligned} \tag{2.7}$$

For $n \geq 1$ and $i \leq n$, we define γ^i_n to be

$$\gamma^i_n = (x_1, \dots, x_{i-1}, x_i \wedge x_{i+1}, x_{i+2}, \dots, x_{n+1}) : [\![n+1]\!] \to [\![n]\!].$$

Finally, for $p \in \Sigma_n$, we let

$$\pi_p = (x_{p^{-1}(1)}, \dots, x_{p^{-1}(n)}) : [\![n]\!] \to [\![n]\!].$$

Note that $\mathcal{Q}$ is isomorphic to the subcategory of $\mathcal{Q}_\Sigma$ generated by the coface and codegeneracy maps $\delta^{i,\varepsilon}_n$ and σ^i_n. The inclusion $\mathcal{Q} \to \mathcal{Q}_\Sigma$ is the strict monoidal functor we described above in terms of forgetting structure.

Proposition 2.10. *The codegeneracy and coface maps satisfy the following relations:*

$$\begin{aligned} &\delta^{j,\eta}\delta^{i,\varepsilon} = \delta^{i,\varepsilon}\delta^{j-1,\eta} \quad \text{if } i < j \\ &\sigma^j\delta^{i,\varepsilon} = \begin{cases} \delta^{i,\varepsilon}\sigma^{j-1} & \text{if } i < j \\ \mathrm{id} & \text{if } i = j \\ \delta^{i-1,\varepsilon}\sigma^j & \text{if } i > j \end{cases} \\ &\sigma^j\sigma^i = \sigma^i\sigma^{j+1} \quad \text{if } i \leq j. \end{aligned} \tag{2.8}$$

The connection maps satisfy the following relations [17]:

$$\begin{aligned} &\gamma^j\gamma^i = \begin{cases} \gamma^i\gamma^{j+1} & \text{if } j > i \\ \gamma^i\gamma^{i+1} & \text{if } j = i \end{cases} \qquad \sigma^j\gamma^i = \begin{cases} \gamma^{i-1}\sigma^j & \text{if } j < i \\ \sigma^i\sigma^i & \text{if } j = i \\ \gamma^i\sigma^{j+1} & \text{if } j > i \end{cases} \\ &\gamma^j\delta^{i,\varepsilon} = \begin{cases} \delta^{i-1,\varepsilon}\gamma^j & \text{if } j < i - 1 \\ \delta^{i,0}\sigma^i & \text{if } j = i - 1, i \text{ and } \varepsilon = 0 \\ \mathrm{id} & \text{if } j = i - 1, i \text{ and } \varepsilon = 1 \\ \delta^{i,\varepsilon}\gamma^{j-1} & \text{if } j > i. \end{cases} \end{aligned} \tag{2.9}$$

Let

$$f_i : \{1 < \cdots < n-1\} \to \{1 < \cdots < n\}$$

denote the unique injective order-preserving map that omits i from its image and

$$g_i : \{1 < \cdots < n+1\} \to \{1 < \cdots < n\}$$

the unique surjective order-preserving map that maps i and i + 1 to i. The cosymmetry maps satisfy the following relations:

$$\begin{aligned}
&\pi_p\pi_q = \pi_{pq} \\
&\pi_p\delta^{i,\varepsilon} = \delta^{p(i),\varepsilon}\pi_{g_{p(i)}pf_i} \\
&\pi_p\gamma^i = \gamma^{p(i)}\pi_q, \quad \textit{where } q(j) = \begin{cases} p(i) & \textit{if } j = i \\ f_{p(i)}(p(g_i(j))) & \textit{otherwise} \end{cases} \\
&\sigma^i\pi_p = \pi_{g_ipf_{p^{-1}(i)}}\sigma^{p^{-1}(i)}.
\end{aligned} \tag{2.10}$$

The proof of Proposition 2.10 is left as an exercise for the reader.

Definition 2.11. Let $\mathcal{Q}_\Sigma^+$ be the subcategory of $\mathcal{Q}_\Sigma$ generated by coface maps $\delta_n^{i,\varepsilon}$ and cosymmetry maps π_p; let $\mathcal{Q}_\Sigma^-$ be the subcategory of $\mathcal{Q}_\Sigma$ generated by codegeneracy maps σ_n^i, connections γ_n^i, and cosymmetry maps π_p.

Proposition 2.12. *Every map f in $\mathcal{Q}_\Sigma$ admits a unique factorization of the form*

$$f = \delta^{i_1,\varepsilon_1}\cdots\delta^{i_n,\varepsilon_n}\gamma^{k_1}\cdots\gamma^{k_r}\pi_p\sigma_\ell^{j_1}\cdots\sigma^{j_m} \tag{2.11}$$

with

$$i_1 > i_2 > \cdots > i_n \qquad j_1 < j_2 < \cdots < j_m \qquad k_1 < k_2 < \cdots < k_r$$

and $p \in \Sigma_\ell$. If $f \in \mathrm{ar}\mathcal{Q}_\Sigma^+$, then $r = m = 0$; if $f \in \mathrm{ar}\mathcal{Q}_\Sigma^-$, then $n = 0$. If $f \in \mathrm{ar}\mathcal{Q}$, then $r = 0$ and $\pi_p = \mathrm{id}$.

Here is an outline of the proof of Proposition 2.12. We read the decomposition (2.11) off of a formal cubical (a, b)-product as follows: the indices $j_1, \ldots, j_m$ correspond to the symbols $x_{j_1}, \ldots, x_{j_m}$ omitted from the formal cubical product. The order of the remaining indices determines π_p uniquely; the list $k_1, \ldots, k_r$ corresponds to the positions in which a concatenation is performed, and the list $i_1, \ldots, i_n$ corresponds to the positions containing a 0 or 1. For example, the $(5, 4)$-product

$$(x_3, 1, x_1 \wedge x_5 \wedge x_2, 0)$$

decomposes as

$$\delta^{4,0}\delta^{2,1}\gamma^2\gamma^3\pi_{(1243)}\sigma^4.$$

This is analogous to the decompositions given by Grandis and Mauri [17, Theorem 8.3]. However, Grandis and Mauri's extended cubical category $\mathbb{K}$ has an additional degeneracy operation given by disjunction $- \vee -$ and some extra relations. Moreover, in $\mathbb{K}$, the operations $\wedge$ and $\vee$ are commutative. As a result, the permutation p in factorizations of the form (2.11) in $\mathbb{K}$ is uniquely determined up to a possibly nontrivial subgroup of Σ_ℓ. One advantage of $\mathbb{K}$ and similar categories is that the vertices functor $\mathbb{K}(\llbracket 0 \rrbracket, -) : \mathbb{K} \to \mathbf{Set}$ is faithful. The analogous vertices functor $\mathcal{Q}_\Sigma(\llbracket 0 \rrbracket, -) : \mathcal{Q}_\Sigma \to \mathbf{Set}$ is not faithful. This is a marked departure from most cubical sites.

Corollary 2.13. *Suppose $f : \llbracket m \rrbracket \to \llbracket n \rrbracket$ is a map in $\mathcal{Q}_\Sigma$. Then f admits a factorization $f = \delta\sigma$ with $\delta \in \mathcal{Q}_\Sigma^+$ and $\sigma \in \mathcal{Q}_\Sigma^-$. Given any other factorization $f = \delta'\sigma'$, the target of σ' and σ agree. Moreover, there is a unique map π such that*

$$\begin{tikzcd}
& \llbracket r \rrbracket \arrow[dd, "\pi"] \arrow[dr, "\delta"] & \\
\llbracket m \rrbracket \arrow[ur, "\sigma"] \arrow[rr, "f"] \arrow[dr, "\sigma'"'] & & \llbracket n \rrbracket \\
& \llbracket r \rrbracket \arrow[ur, "\delta'"'] &
\end{tikzcd}$$

commutes; in fact π is always a cosymmetry isomorphism.

Corollary 2.14. *Let $\mathcal{F}$ denote the category with objects $\llbracket 0 \rrbracket, \llbracket 1 \rrbracket, \ldots$ and with maps freely generated by the symbols $\delta_n^{i,\varepsilon}$, σ_n^i, γ_n^i, and π_p (where i, n, and p vary appropriately) subject to the relations in Proposition 2.10. The functor $F : \mathcal{F} \to \mathcal{Q}_\Sigma$ mapping $\delta^{i,\varepsilon}$ to $\delta^{i,\varepsilon}$, etc. is an isomorphism of categories.*

To see Corollary 2.14, we use Proposition 2.12 twice. The functor F is an isomorphism on objects by definition. The existence of the decomposition (2.11) in Proposition 2.12 shows that F is full. The identities in Proposition 2.10 let us rewrite any word in the symbols $\delta_n^{i,\varepsilon}$, σ_n^i, γ_n^i, and π_p in the canonical form (2.11) in $\mathcal{F}$ (to prove this, induct on the length of words). The uniqueness of this canonical form in $\mathcal{Q}_\Sigma$ shows that F is faithful.

2.4. $\mathscr{Q}_\Sigma$ parameterizes cubical monoids

We close this section with a proof of the provisional definition we first gave of $\mathscr{Q}_\Sigma$. Note that $[\![1]\!]$ forms a cubical monoid in $\mathscr{Q}_\Sigma$ with units $d_\varepsilon = \delta_0^{1,\varepsilon} : [\![0]\!] \to [\![1]\!]$, multiplication $\mu = \gamma_1^1$, and augmentation $s = \sigma_0^1$. Hence if $G : \mathscr{Q}_\Sigma \to \mathscr{C}$ is a strong symmetric monoidal functor to some symmetric monoidal category $\mathscr{C}$, then $G([\![1]\!])$ together with all the attendant structure maps is a cubical monoid in $\mathscr{C}$.

Proposition 2.15. *Suppose $\mathscr{C}$ is a symmetric monoidal category. Let $\mathrm{Fun}^{\otimes}(\mathscr{Q}_\Sigma, \mathscr{C})$ denote the category of strong symmetric monoidal functors $\mathscr{Q}_\Sigma \to \mathscr{C}$ and natural transformations. Evaluation at $[\![1]\!]$ defines an equivalence of categories*

$$F : \mathrm{Fun}^{\otimes}(\mathscr{Q}_\Sigma, \mathscr{C}) \to \mathbf{qMon}(\mathscr{C}).$$

Proof. We give an outline of the proof. Suppose I is a cubical monoid in $\mathscr{C}$ with unit maps d_0 and d_1, multiplication μ, and augmentation s. We attempt to define a strong symmetric monoidal functor $G(I) : \mathscr{Q}_\Sigma \to \mathscr{C}$ as follows: set $G(I)([\![n]\!]) = I^{\otimes n}$. We define

$$G(I)(\delta_n^{i,\varepsilon}) = \mathrm{id}_{I^{i-1}} \otimes d_\varepsilon \otimes \mathrm{id}_{I^{n-i+1}}$$
$$G(I)(\sigma_n^{i}) = \mathrm{id}_{I^{i-1}} \otimes s \otimes \mathrm{id}_{I^{n-i+1}}$$
$$G(I)(\gamma_n^{i}) = \mathrm{id}_{I^{i-1}} \otimes \mu \otimes \mathrm{id}_{I^{n-i}}$$

and define $G(I)(\pi_p)$ by making use of the symmetry in $\mathscr{C}$ (so, e.g., $G(I)(\pi_{(12)})$ is the interchange map $I \otimes I \to I \otimes I$). To extend $G(I)$ to a functor, we use the factorization in Proposition 2.12. However, we need to check that $G(I)$, so defined, is a functor—by Corollary 2.14, we need to check that $G(I)$ satisfies the relations in Proposition 2.10. Since I is a cubical monoid, $s : I \to e$ is a monoid map. This gives the relation

$$G(I)(\sigma^i) \circ G(I)(\gamma^i) = G(I)(\sigma^i) \circ G(I)(\sigma^i).$$

The remaining relations are left as an exercise. The functor G gives an inverse equivalence to F. $\square$

3. The structure of extended cubical sets

In this section, we present some machinery that allows us to decompose cubical sets and symmetric cubical sets as colimits of their skeleta. We first need a workable definition of skeleton. There is a general theory due to Cisinski of skeletal decompositions generalizing the classical theory for simplicial sets in [13,15]—see [9, Chapître 8]. Berger and Moerdijk also discuss an apparatus for skeletal decomposition in [2]—the theory of Eilenberg–Zilber categories—which we apply to $\mathscr{Q}$ and $\mathscr{Q}_\Sigma$. Before we get started, we will introduce some notation.

3.1. Notation

Suppose $\mathscr{C}$ is a category. We write $\widehat{\mathscr{C}}$ for the category of presheaves of sets on $\mathscr{C}$ and $[-] : \mathscr{C} \to \widehat{\mathscr{C}}$ for the Yoneda embedding: $[X]$ is the functor $Y \mapsto \mathscr{C}(Y, X)$. We abbreviate adjunctions

$$F : \mathscr{C} \rightleftarrows \mathscr{D} : G$$

with the shorthand $F \dashv G$.

3.2. Eilenberg–Zilber categories and decompositions

Suppose X is a simplicial set. Given any n-simplex $f : \Delta[n] \to X$, we may take a factorization

$$\begin{tikzcd} \Delta[n] \arrow[rr, "f"] \arrow[dr, "s"'] & & X \\ & \Delta[r] \arrow[ur, "g"'] & \end{tikzcd} \tag{3.1}$$

so that $s : \underline{n} \to \underline{r}$ is an epimorphism (i.e., a degeneracy map) and r is minimal among all such factorizations. Of course, the minimality of r implies that the simplex g is *nondegenerate*—it does not factor through another degeneracy. One feature of the combinatorics of Δ is that this factorization is unique: if $f = g's'$ is another factorization with g' nondegenerate and s' a degeneracy map, then $g' = g$ and $s' = s$. This seemingly innocuous observation allows us to identify the m-skeleton of X (usually given as the counit of the left Kan extension/restriction adjunction along $\Delta_{\le m} \to \Delta$) as the subpresheaf of X whose n-simplices are those n-simplices f so that $r \le m$ in the Eilenberg–Zilber decomposition (3.1) of f. A simple induction argument then shows that the maps $\partial\Delta[n] \to \Delta[n]$, $n \ge 0$ are a *cellular model* for $s\mathbf{Set}$: the monomorphisms in $s\mathbf{Set}$ comprise the smallest class of arrows containing $\partial\Delta[n] \to \Delta[n]$, $n \ge 0$, closed under cobase change, transfinite composition, and retract.

These sorts of arguments also work in $q\mathbf{Set}$ (as we will see below), but not in $q_\Sigma\mathbf{Set}$ without some modification. The identity map $\square^n_\Sigma \to \square^n_\Sigma$ is nondegenerate, in the sense that it does not factor through any non-invertible degeneracies, but any symmetry π yields a factorization $\pi^{-1}\pi$. Also, the maps $\iota_n : \partial\square^n_\Sigma \to \square^n_\Sigma$ do not comprise a cellular model for

q_{Σ}**Set**: there is no way to form, e.g., the quotient $(\Box^2_{\Sigma})/\Sigma_2$ of $\Box^2_{\Sigma}$ by the action of Σ_2 with iterated cobase changes, transfinite compositions, and retracts of the maps ι_n. As it turns out, these are the only two complications that arise when we try to apply Gabriel and Zisman's theory to $\mathcal{Q}_{\Sigma}$. We need to replace uniqueness with a properly categorical notion—contractible groupoids—and we need to keep track of the action of $\mathrm{Aut}(\Box^n_{\Sigma})$ on $\Box^n_{\Sigma}$. The appropriate generalization of Δ is the notion of an *Eilenberg–Zilber* category, due to Berger and Moerdijk [2] and Cisinski [9], which we introduce below.

Definition 3.1. Suppose $\mathscr{C}$ is a category and $\mathscr{I}$ a small category. Suppose further that $X : \mathscr{I} \to \mathscr{C}$ is a diagram and $X \to c_Y$ is a cocone on X (here c_Y is the constant $\mathscr{I}$-diagram on Y). We say $X \to c_Y$ is an *absolute colimit* if $FX \to c_{FY}$ is a colimit for all functors $F : \mathscr{C} \to \mathscr{D}$.

Split coequalizers are examples of absolute colimits [26]. In Definition 3.1, it is necessary and sufficient to check that $[X] \to c_{[Y]}$ is a colimit (Paré [30]).

Definition 3.2 (*[2, Definition 6.6]*)**.** An *Eilenberg–Zilber category* (briefly EZ *category*) is a small category $\mathscr{R}$ together with a degree function $\deg : \mathrm{ob}\mathscr{R} \to \mathbf{Z}_{\geq 0}$ such that

(EZ1) Monomorphisms preserve the degree if and only if they are invertible; they raise the degree if and only if they are non-invertible.
(EZ2) Every morphism factors as a split epimorphism followed by a monomorphism.
(EZ3) Suppose

$$s_1 \xleftarrow{\sigma_1} r \xrightarrow{\sigma_2} s_2$$

is a pair of split epimorphisms. Then there is an absolute pushout square

$$\begin{tikzcd} r \arrow[r, "\sigma_2"] \arrow[d, "\sigma_1"'] & s_2 \arrow[d, "\tau_2"] \\ s_1 \arrow[r, "\tau_1"'] & t \end{tikzcd}$$

in $\mathscr{R}$ in which τ_1 and τ_2 are split epimorphisms.

We define $\mathscr{R}^+$ (respectively $\mathscr{R}^-$) to be the subcatetegory of $\mathscr{R}$ with the same objects and with arrows the split epimorphisms (respectively with arrows the monomorphisms).

Suppose $\mathscr{R}$ is an EZ category whose only isomorphisms are identity maps. The factorization provided by EZ2 is then unique by EZ3. Moreover, since the section of a split epimorphism is monic, non-identity split epimorphisms lower degree. In this special case, $\mathscr{R}$ is an example of a *Reedy category:*

Definition 3.3. Suppose $\mathscr{C}$ is a category and $\mathscr{D}$ a subcategory of $\mathscr{C}$; we say $\mathscr{D}$ is *wide* if $\mathrm{ob}\mathscr{D} = \mathrm{ob}\mathscr{C}$. A *Reedy category* [12,19,20] is a small category $\mathscr{R}$ together with a degree function $\deg : \mathrm{ob}\mathscr{R} \to \mathbf{Z}_{\geq 0}$ and two wide subcategories $\mathscr{R}^+$ and $\mathscr{R}^-$ so that

(R1) Non-identity morphisms in $\mathscr{R}^+$ raise the degree; non-identity morphisms in $\mathscr{R}^-$ lower the degree.
(R2) Every morphism $f \in \mathrm{ar}\mathscr{R}$ factors uniquely as $f = gh$ with $g \in \mathrm{ar}\mathscr{R}^+$ and $h \in \mathrm{ar}\mathscr{R}^-$.

Not all Reedy categories are EZ. As one might expect, Δ is both EZ and Reedy. The main result of this section is the following:

Proposition 3.4. *The categories $\mathcal{Q}$ and $\mathcal{Q}_{\Sigma}$ are* EZ *categories.*

The proof of this, especially the verification of EZ3, is rather technical and we postpone it to the end of the section. Before we get to it, we continue with a discussion of the properties of EZ categories.

3.3. Skeleta, coskeleta, and cellular models

Definition 3.5. Let $\mathscr{R}$ be an EZ category and suppose $X \in \widehat{\mathscr{R}}$. We say a section $x \in X_r$ is *degenerate* if there is a map $\sigma : r \to s$ in $\mathscr{R}^-$ and $y \in X_s$ so that $\sigma^* y = x$ and $\deg s < \deg r$.

Proposition 3.6 (*[2, Proposition 6.7]*)**.** *Let $\mathscr{R}$ be an* EZ *category.*

(1) *Suppose $X \in \widehat{\mathscr{R}}$. Let $x \in X_r$, $r \in \mathrm{ob}\mathscr{R}$. The category of factorizations*

$$\begin{tikzcd} {[r]} \arrow[rr, "x"] \arrow[dr, "\sigma_*"'] & & X \\ & {[s]} \arrow[ur, "y"'] & \end{tikzcd}$$

with $\sigma \in \mathscr{R}^-$ and y nondegenerate is a contractible groupoid.

(2) *If $f : r \to s$ is an arrow in $\mathscr{R}$, the category of factorizations*

$$\begin{tikzcd} r \arrow[rr, "f"] \arrow[dr, "f^-"'] & & s \\ & t \arrow[ur, "f^+"'] & \end{tikzcd}$$

with f^- a split epimorphism and f^+ a monomorphism is a contractible groupoid.

Following [2,15], we call any such factorization an EZ *decomposition* of x. The Proposition implies in particular that EZ decompositions exist.

Definition 3.7. Suppose $\mathscr{R}$ is an EZ category and $n \geq -1$. Write $\mathscr{R}_{\leq n}$ for the full subcategory of $\mathscr{R}$ with objects those of degree at most n. The inclusion $j_n : \mathscr{R}_{\leq n} \to \mathscr{R}$ yields adjunctions

$$\widehat{\mathscr{R}_{\leq n}} \underset{(j_n)^*}{\overset{(j_n)_!}{\rightleftarrows}} \widehat{\mathscr{R}} \underset{(j_n)_*}{\overset{(j_n)^*}{\rightleftarrows}} \widehat{\mathscr{R}_{\leq n}} \tag{3.2}$$

given by left and right Kan extension. We define the *n-skeleton* and *n-coskeleton* of $X \in \widehat{\mathscr{R}}$ to be

$$\mathrm{sk}_n X = (j_n)_!(j_n)^* X \quad \text{and} \quad \mathrm{ck}_n X = (j_n)_*(j_n)^* X$$

respectively. The counit and unit of the adjunctions in (3.2) yield natural maps

$$\mathrm{sk}_n X \longrightarrow X \longrightarrow \mathrm{ck}_n X.$$

We say X is *n-skeletal* if $\mathrm{sk}_n X \to X$ is an isomorphism and *n-coskeletal* if $X \to \mathrm{ck}_n X$ is an isomorphism.

In a precise sense, the n-skeleton of $X \in \widehat{\mathscr{R}}$, $\mathscr{R}$ an EZ category, is the subpresheaf generated by the non-degenerate sections of X of degree at most n.

Proposition 3.8 (*[2]*). *Suppose $\mathscr{R}$ is an* EZ *category and $X \in \widehat{\mathscr{R}}$. The map $\mathrm{sk}_n X \to X$ is a monomorphism; its image in X_r, $r \in \mathrm{ob}\mathscr{R}$ is the set of sections*

$$\left\{f \in X_r \;\middle|\; f \text{ has a factorization } [r] \to [s] \to X \text{ with } \deg s \leq n\right\}.$$

Definition 3.9. Suppose $\mathscr{R}$ is an EZ category and $r \in \mathrm{ob}\mathscr{R}$. We define the *boundary* $\partial[r]$ of $[r]$ to be the $(n-1)$-skeleton of $[r]$, where $\deg r = n$.

Proposition 3.10. *Suppose $\mathscr{R}$ is an* EZ *category whose only isomorphisms are identity maps. Suppose $X \in \mathscr{R}$ and $n \geq 0$. Let S be the set of maps $f : [r] \to X$ with $\deg r = n$ and f nondegenerate. The square*

$$\begin{tikzcd} \coprod_{f:[r]\to X\in S} \partial[r] \arrow[r] \arrow[d] & \mathrm{sk}_{n-1}X \arrow[d] \\ \coprod_{f:[r]\to X\in S} [r] \arrow[r] & \mathrm{sk}_n X \end{tikzcd} \tag{3.3}$$

is a pushout.

Proof. This proof is a straightforward generalization of [15, Section II.3.8]. Since every object in (3.3) is n-skeletal, it is sufficient to check that the restriction of (3.3) to $\mathscr{R}_{\leq n}$ is a pushout square. In a presheaf topos, pushouts are computed pointwise, so it is sufficient to prove that the square (3.3) is a pushout after evaluation at s for all $s \in \mathrm{ob}\mathscr{R}_{\leq n}$. If $\deg s < n$ and $\deg r = n$, the maps

$$\big(\partial[r]\big)_s \to [r]_s \quad \text{and} \quad (\mathrm{sk}_{n-1}X)_s \to (\mathrm{sk}_n X)_s,$$

are isomorphisms. Thus we are reduced to checking that

$$\begin{tikzcd} \coprod_{f:[r]\to X\in S} \big(\partial[r]\big)_s \arrow[r] \arrow[d] & (\mathrm{sk}_{n-1}X)_s \arrow[d] \\ \coprod_{f:[r]\to X\in S} [r]_s \arrow[r] & (\mathrm{sk}_n X)_s \end{tikzcd} \tag{3.4}$$

is a pushout when $\deg s = n$.

Suppose $\deg s = n$. The complement of $(\mathrm{sk}_{n-1}X)_s$ in $(\mathrm{sk}_n X)_s$ is the set of all nondegenrate s-simplices $[s] \to X$. Since $\mathscr{R}$ has no nontrivial isomorphisms, if $r \neq s$ has degree n, each map $s \to r$ factors through an object of lower degree, so

$$\big(\partial[r]\big)_s \to [r]_s$$

is an isomorphism. On the other hand, the complement of the image of

$$\big(\partial[s]\big)_s \to [s]_s$$

is the identity map $s \to s$, so the complement of the image of

$$\coprod_{f:[r]\to X\in S} \big(\partial[r]\big)_s \to \coprod_{f:[r]\to X\in S} [r]_s$$

is the set of nondegenerate s-simplices $[s] \to X$. Hence (3.4) is a pushout. □

We can reinterpret Proposition 3.10 as a statement about saturated classes of maps. We first introduce the following definition, using Cisinski's terminology [9]:

Definition 3.11. Suppose $\mathscr{R}$ is a small category. We say that a set of arrows $S \subseteq \mathrm{ar}\widehat{\mathscr{R}}$ is a *cellular model* for $\widehat{\mathscr{R}}$ if $\mathrm{Cell}S = \mathsf{mono}$. Here, $\mathrm{Cell}S$ is the closure of the set S under transfinite composition, cobase change, coproduct, and retract.

Any topos has a cellular model [8,1]. In the case of a presheaf topos, all inclusions of subobjects of (regular) quotients of representables form a cellular model. In EZ categories, we have the expected simplification:

Corollary 3.12. *Under the assumptions of Proposition 3.10, the arrows $\partial[r] \to [r]$, $r \in \mathrm{ob}\mathscr{R}$, comprise a cellular model for $\widehat{\mathscr{R}}$.*

This corollary may seem slightly weaker than Proposition 3.10 because it does not say anything about the dimension of the attaching maps (bringing to mind the distinction between cellular and cw complexes in **Top**). In $\widehat{\mathscr{R}}$ however, every map $\partial[r] \to X$ automatically factors through $\mathrm{sk}_{\deg r-1}X \to X$.

Proof of Corollary 3.12. Let C temporarily denote the class of arrows

$$\mathrm{Cell}\{\partial[r] \to [r] \mid r \in \mathrm{ob}\mathscr{R}\}.$$

Since $\widehat{\mathscr{R}}$ is a topos, $C \subseteq \mathsf{mono}$. Recall that $\mathrm{sk}_{-1}A = \mathrm{sk}_{-1}B = \emptyset$. Suppose $f : A \to B$ is a monomorphism in $\widehat{\mathscr{R}}$. Let $\mathrm{sk}_n f$ be the pushout $\mathrm{sk}_n B \coprod_{\mathrm{sk}_n A} A$ and let $p_n : \mathrm{sk}_n f \to B$ be the corner map. Note that the square

$$\begin{tikzcd} \mathrm{sk}_n A \arrow[r, "\mathrm{sk}_n f"] \arrow[d] & \mathrm{sk}_n B \arrow[d] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

is a pullback. Since $\widehat{\mathscr{R}}$ is a topos, $\mathrm{sk}_n f$ is the effective union of $\mathrm{sk}_n B$ and A in $\mathrm{sk}_n f$ and p_n is a monomorphism. The square

$$\begin{tikzcd} \mathrm{sk}_n B \arrow[r] \arrow[d] & \mathrm{sk}_{n+1} B \arrow[d] \\ \mathrm{sk}_n f \arrow[r] & \mathrm{sk}_{n+1} f \end{tikzcd}$$

is a pushout, so $\mathrm{sk}_n f \to \mathrm{sk}_{n+1} f$ is in C. Now $\mathrm{colim}_n \mathrm{sk}_n f \to B$ is an isomorphism. Since $\mathrm{sk}_{-1} f = A$, we have realized f as a transfinite composition of maps in C, so $f \in C$. Hence $C = \mathsf{mono}$. □

Note that Proposition 3.10 is false if we allow objects in $\mathscr{R}$ to have nontrivial automorphisms. An easy example is the one-object category associated to a group G. This is an EZ category with $\deg * = 0$. An object $X \in \widehat{G}$ is a right G-set; were Proposition 3.10 true, it would imply that all $X \in \widehat{G}$ are free as G-sets. We now prove a generalization of Proposition 3.10 for categories $\mathscr{R}$ containing nontrivial isomorphisms.

Definition 3.13. Suppose $\mathscr{R}$ is an EZ category, $X \in \widehat{\mathscr{R}}$, and $f : [r] \to X$ is a nondegenerate r-simplex of X. Note that $\mathrm{Aut}(r)$ acts on X_r on the right. The *isotropy of f*, denoted $\mathrm{Stab}(f)$, is the stabilizer of $f \in X_r$ in $\mathrm{Aut}(r)$, i.e., the subgroup of $g \in \mathrm{Aut}(r)$ with $g^* f = f$.

In the following, note that the left action of $\mathrm{Aut}(r)$ on $[r]$ restricts to an action on $\partial[r]$. If $H \leqslant \mathrm{Aut}(r)$, then $(\partial[r])/H \to [r]/H$ is a monomorphism (here X/H denotes the H-orbits of X).

Proposition 3.14. *Suppose $\mathscr{R}$ is a skeletal* EZ *category, i.e., two objects are isomorphic if and only if they are equal. Let $n \geq 0$ and let S be a set of isomorphism classes of $f : [r] \to X$ with $\deg r = n$ and f nondegenerate. Then the square*

$$\begin{tikzcd} \coprod_{f:[r]\to X\in S} \partial[r]/\mathrm{Stab}\, f \arrow[r] \arrow[d] & \mathrm{sk}_{n-1}X \arrow[d] \\ \coprod_{f:[r]\to X\in S} [r]/\mathrm{Stab}\, f \arrow[r] & \mathrm{sk}_n X \end{tikzcd}$$

is a pushout.

Proof. Note that since sk_n is cocontinuous, if $Y \in \mathscr{R}$ is n-skeletal and a group G acts on Y, Y/G is n-skeletal as well. Just as in the proof of Proposition 3.10, it is sufficient to check that

$$\begin{tikzcd} \coprod_{f:[r]\to X\in S} \left(\partial[r]/\mathrm{Stab}\, f\right)_s \arrow[r] \arrow[d] & (\mathrm{sk}_{n-1}X)_s \arrow[d] \\ \coprod_{f:[r]\to X\in S} \left([r]/\mathrm{Stab}\, f\right)_s \arrow[r] & (\mathrm{sk}_n X)_s \end{tikzcd} \tag{3.5}$$

is a pushout when $\deg s = n$. The complement of $(\mathrm{sk}_{n-1}X)_s$ in $(\mathrm{sk}_n X)_s$ is the right $\mathrm{Aut}(s)$-set of all nondegenerate s-simplices $f : [s] \to X$.

Now we have two possibilities. Suppose $r \neq s$ has degree n. If $f : [r] \to X$ is a nondegenerate r-simplex and $r \neq s$,

$$\left(\partial[r]/\mathrm{Stab}\, f\right)_s \to \left([r]/\mathrm{Stab}\, f\right)_s$$

is an isomorphism: any map $s \to r$ must factor through an object of lower degree since a degree-preserving map $s \to r$ is necessarily an isomorphism (recall that we have assumed $\mathscr{R}$ is skeltal). On the other hand, if $H \leqslant \mathrm{Aut}(s)$, the complement of the image of

$$\left(\partial[s]/H\right)_s \to \left([s]/H\right)_s$$

is the right $\mathrm{Aut}(s)$-set $\mathrm{Aut}(s)/H$. Thus the complement of the image of

$$\coprod_{f:[r]\to X\in S} \left(\partial[r]/\mathrm{Stab}\, f\right)_s \to \coprod_{f:[r]\to X\in S} \left([r]/\mathrm{Stab}\, f\right)_s$$

is the right $\mathrm{Aut}(s)$-set

$$\coprod_{g:[s]\to X\in S} \mathrm{Aut}(s)/\mathrm{Stab}(g).$$

This decomposition maps isomorphically onto $(\mathrm{sk}_n X)_s \setminus (\mathrm{sk}_{n-1}X)_s$ via the map sending the coset $\mathrm{Stab}(g)$ to g. Hence (3.5) is a pushout.

Corollary 3.15. *Under the assumptions of Proposition 3.14, the set*

$$\left\{\partial[r]/H \to [r]/H \;\middle|\; r \in \mathrm{ob}\mathscr{R} \text{ and } H \leqslant \mathrm{Aut}\, r\right\}$$

is a cellular model for $\widehat{\mathscr{R}}$.

As we will see below, $\mathscr{Q}$ and $\mathscr{Q}_\Sigma$ are EZ categories, so we obtain the following cellular models. We write $\square^n$ and $\square^n_\Sigma$ for the representables $\mathscr{Q}(-, [\![n]\!])$ and $\mathscr{Q}_\Sigma(-, [\![n]\!])$ respectively.

Proposition 3.16. *The sets*

$$I = \left\{\partial\square^n \to \square^n \;\middle|\; n \geq 0\right\} \tag{3.6}$$

$$I_\Sigma = \left\{\partial\square^n_\Sigma/H \to \square^n_\Sigma/H \;\middle|\; n \geq 0 \text{ and } H \leqslant \Sigma_n\right\}$$

*are cellular models for q**Set** and q_Σ**Set**, respectively.*

3.4. Comparing skeletal filtrations

In this section we prove a base-change theorem that allows us to compare skeleta of cubical and extended cubical sets. We begin with a slightly modified definition from [9, Chapître 8]:

Definition 3.17. Suppose $i : \mathscr{R} \to \mathscr{S}$ is a functor. We say that i is a *thickening* if

(1) i is an isomorphism on objects.
(2) For all $r, r' \in \mathrm{ob}\mathscr{R}$, the map
$$\mathrm{Aut}_{\mathscr{S}}(ir) \times \mathscr{R}(r, r') \to \mathscr{S}(ir, ir')$$
is a bijection of sets.

Crossed Δ-modules, and more generally crossed $\mathscr{R}$-modules for a Reedy category $\mathscr{R}$, are examples of thickenings [2,14]. Note that $\mathscr{Q} \to \mathscr{Q}_\Sigma$ is not a thickening, but $\mathscr{Q}^+ \to \mathscr{Q}^+_\Sigma$ is a thickening by Proposition 2.12. We start with a simple observation:

Lemma 3.18. *Suppose $i : \mathscr{R} \to \mathscr{S}$ is a thickening and*

$$\begin{tikzcd} & ir_2 \arrow[dr, "if"] & \\ ir_1 \arrow[ur, "\sigma"] \arrow[rr, "ig"'] & & ir_3 \end{tikzcd}$$

is a diagram in which σ is an arrow in $\mathscr{S}$. Then σ is in the image of i and the triangle may be lifted to one in $\mathscr{R}$.

Proof. Since i is a thickening, there is a (unique) factorization of σ as a composition $(ih) \circ \tau$, where $\tau \in \mathrm{Aut}_{\mathscr{S}}(ir_1)$ and h is a map $r_1 \to r_2$. Then $i(gh) \circ \tau = if$, so by the uniqueness of factorizations of this form, $\tau = \mathrm{id}_{ir_1}$ and hence $\sigma = ih$. $\square$

Proposition 3.19. *Suppose $i : \mathscr{R} \to \mathscr{S}$ is a functor between* EZ *categories $\mathscr{R}$ and $\mathscr{S}$ so that i preserves degree. Then i preserves monomorphisms; suppose that, moreover, the resulting functor $i^+ : \mathscr{R}^+ \to \mathscr{S}^+$ is a thickening. The natural base-change transformation $i_! j^* \to j^* i_!$ of functors $\widehat{\mathscr{R}} \to \widehat{\mathscr{S}_{\leq n}}$ induced by the square of functors*

$$\begin{tikzcd} \mathscr{R}_{\leq n} \arrow[r, "j"] \arrow[d, "i"'] & \mathscr{R} \arrow[d, "i"] \\ \mathscr{S}_{\leq n} \arrow[r, "j"'] & \mathscr{S} \end{tikzcd}$$

is a natural isomorphism.

Proof. The functors $i_!$ and j^* preserve colimits, so it is sufficient to check that $i_! j^* \to j^* i_!$ is an isomorphism on all representables $[r] \in \widehat{\mathscr{R}}$, $r \in \mathrm{ob}\mathscr{R}$. Suppose $r \in \mathrm{ob}\mathscr{R}$ and $s \in \mathrm{ob}\mathscr{S}_{\leq n}$. Note that $j^*[r] \cong \mathrm{colim}_{r'}[r']$ where the colimit is taken over all r' with $\deg r' < n$, so $i_! j^*[r] \cong \mathrm{colim}_{r'}[ir']$. As a result, on the level of sets, the map $\varphi : \left(i_! j^*[r]\right)_s \to \left(j^* i_![r]\right)_s$ is given by
$$\operatorname*{colim}_{\substack{r' \to r \\ \deg r' \leq n}} \left\{s \to ir' \in \mathrm{ar}\mathscr{S}\right\} \to \{s \to ir \in \mathrm{ar}\mathscr{S}\}.$$

Now suppose $g : s \to ir$ is a map in $\mathscr{S}$. Since $\mathscr{S}$ is an EZ category and i^+ is a thickening, there is a factorization

$$\begin{tikzcd} & ir' \arrow[dr, "ig^+"] & \\ s \arrow[ur, "g^-"] \arrow[rr, "g"'] & & ir \end{tikzcd}$$

in which g^- is a split epimorphism in $\mathscr{S}$ and g^+ is a monomorphism in $\mathscr{R}$. Since $\deg s \leq n$, $\deg r' \leq n$ as well. Hence φ is a surjection. We can assume, moreover, that the degree of r' is minimal among all such factorizations (in fact, there is only one possible degree).

Now suppose we have maps $h : s \to ir_1$ in $\mathscr{S}$ and $\ell : r_1 \to r$ in $\mathscr{R}$ so that $\deg r_1 \leq n$ and $i\ell \circ h = g$. We must show that the pair (ℓ, h) is identified with (g^+, g^-) in the colimit in the source of φ. By repeated factorizations, we can produce a diagram

$$\begin{tikzcd} s \arrow[r, "h"] \arrow[dr, "h^-"'] & ir_1 \arrow[rr, "i\ell"] \arrow[dr, "i\ell^-"] & & ir \\ & ir_3 \arrow[r, "ih^+"'] & ir_2 \arrow[ur, "i\ell^+"'] & \end{tikzcd}$$

in which h^- is a split epimorphism in $\mathscr{S}$, ℓ^- is a split epimorphism in $\mathscr{R}$, and both ℓ^+ and h^+ are monomorphisms in $\mathscr{R}$. The pairs

$$(\ell, h) \qquad (\ell^+, h \circ i\ell^-) \qquad (\ell^+ h^+, h^-)$$

are identified in the colimit in the source of φ. (Note that $\deg r_3 \leq \deg r_1$.) Without loss of generality, then, we can assume that h is a split epimorphism and ℓ is a monomorphism. But split epi-monic factorizations in $\mathscr{S}$ are essentially unique (Proposition 3.6), so there is an isomorphism σ making

$$\begin{tikzcd} & ir' \arrow[dd, "\sigma"] \arrow[dr, "ig^+"] & \\ s \arrow[ur, "g^-"] \arrow[rr, "g"] \arrow[dr, "h"'] & & ir \\ & ir_1 \arrow[ur, "i\ell"'] & \end{tikzcd}$$

commute. Since i^+ is a thickening, the map σ must be in the image of i (Lemma 3.18), so (ℓ, h) and (g^+, g^-) are identified in the colimit in the source of φ. Hence φ is a bijection of sets. □

Corollary 3.20. *Suppose $i : \mathscr{R} \to \mathscr{S}$ is a functor between* EZ *categories satisfying the assumptions of Proposition 3.19. If $X \in \widehat{\mathscr{R}}$, then there is a natural isomorphism $\mathrm{sk}_n i_! X \to i_! \mathrm{sk}_n X$.*

Proof. With the notation of Proposition 3.19, there is a natural isomorphism $i_! j^* X \to j^* i_! X$. Now apply the functor $j_!$; we obtain a natural isomorphism $j_! i_! j^* X \to j_! j^* i_! X$. Since $j_! i_! \cong i_! j_!$, we obtain a natural isomorphism $i_! j_! j^* X \to j_! j^* i_! X$. □

3.5. The cubical sites

In the remainder of this section, we will prove Proposition 3.4. We begin with the definition $\deg[\![n]\!] = n$. Axioms EZ1 and EZ2 are routine; the difficult axiom to verify will be EZ3.

Lemma 3.21. *All epimorphisms of $\mathscr{Q}$ and $\mathscr{Q}_\Sigma$ are split. The epimorphisms of $\mathscr{Q}$ (respectively $\mathscr{Q}_\Sigma$) correspond to the arrows of $\mathscr{Q}^-$ (respectively $\mathscr{Q}_\Sigma^-$).*

Proof. Both the degeneracies σ^i and γ^i have sections, so they are categorical epimorphisms. Since the cosymmetry maps π_p are isomorphisms, we may conclude that the arrows of $\mathscr{Q}_\Sigma^-$ and $\mathscr{Q}^-$ are split epimorphisms in $\mathscr{Q}_\Sigma$ and $\mathscr{Q}$, respectively.

Suppose f is an arrow in $\mathscr{Q}_\Sigma$. We may factor f as

$$f = \delta^{i_1,\varepsilon_1} \cdots \delta^{i_n,\varepsilon_n} s$$

with $s \in \mathrm{ar}\mathscr{Q}_\Sigma^-$. Suppose $n > 0$. Then

$$f = \delta^{i_1,\varepsilon_1} \sigma^{i_1} f$$

by the relations in Proposition 2.10. However, $\delta^{i_1,\varepsilon_1}\sigma^{i_1} \neq \mathrm{id}$, so f is not an epimorphism. Hence the epimorphisms of $\mathscr{Q}_\Sigma$ are precisely the maps of $\mathscr{Q}_\Sigma^-$, which are all split. The proof for $\mathscr{Q}$ is identical. □

Definition 3.22. Suppose

$$\begin{tikzcd} A \arrow[r, "a_1"] \arrow[d, "a_2"'] & B \arrow[d, "p_2"] \\ C \arrow[r, "p_1"'] & P \end{tikzcd} \tag{3.7}$$

is a commutative square in a category $\mathscr{C}$. If there exist maps

$$d_0 : P \to B \qquad d_1 : C \to A \qquad d_1' : B \to A \qquad d_2' : B \to A$$

so that

$$\begin{aligned} d_0 p_1 &= a_1 d_1 & d_0 p_2 &= a_1 d_1' \\ a_2 d_1 &= \mathrm{id}_C & a_2 d_1' &= a_2 d_2' \\ p_2 d_0 &= \mathrm{id}_P & a_1 d_2' &= \mathrm{id}_B \end{aligned}$$

then we call (3.7) a *split pushout*.

Lemma 3.23. *Split pushouts are absolute pushouts.*

Proof. This is an example of a general criterion by Paré, who classifies all absolute pushouts in [30, Proposition 5.5] (the cited paper also classifies all absolute colimits in general). It is sufficient to check that a split pushout of the shape (3.7) is a pushout square, since split pushouts are manifestly preserved by all functors. Suppose $f : B \to X$ and $g : C \to X$ are given so that $fa_1 = ga_2$. Define $h : P \to X$ to be $h = fd_0$. Then

$$hp_1 = fd_0p_1 = fa_1d_1 = ga_2d_1 = g$$

and

$$hp_2 = fd_0p_2 = fa_1d_1' = ga_2d_1' = ga_2d_2' = fa_1d_2' = f.$$

Since p_2 is a split epimorphism, h is the unique map making

$$\begin{tikzcd} A \arrow[r, "a_1"] \arrow[d, "a_2"'] & B \arrow[d, "p_2"] \arrow[ddr, bend left, "f"] & \\ C \arrow[r, "p_1"'] \arrow[drr, bend right, "g"'] & P \arrow[dr, "h"] & \\ & & X \end{tikzcd}$$

commute. $\square$

In all the split pushouts we compute below, we will always set $d_1' = d_2'd_0p_2$. This reduces the relations that we need to verify to the following five:

$$\begin{aligned} a_2d_1 &= \mathrm{id}_C & d_0p_1 &= a_1d_1 \\ p_2d_0 &= \mathrm{id}_P & a_2d_2'd_0p_2 &= a_2d_2' \\ a_1d_2' &= \mathrm{id}_B. \end{aligned}$$

Lemma 3.24. *The diagram*

$$[\![n-1]\!] \xleftarrow{\sigma^i} [\![n]\!] \xrightarrow{\sigma^j} [\![n-1]\!] \tag{3.8}$$

has an absolute pushout

$$\begin{tikzcd} {[\![n]\!]} \arrow[r, "\sigma^i"] \arrow[d, "\sigma^j"'] & {[\![n-1]\!]} \arrow[d, "\tau_2"] \\ {[\![n-1]\!]} \arrow[r, "\tau_1"'] & {[\![\ell]\!]} \end{tikzcd}$$

in $\mathcal{Q}$ with both τ_1 and τ_2 in $\mathcal{Q}^-$ and $n-2 \le \ell \le n-1$.

Proposition 3.25. *The category $\mathcal{Q}$ satisfies axiom EZ3 of Definition 3.2.*

Corollary 3.26. *The diagram*

$$[\![n-1]\!] \xleftarrow{\sigma^i} [\![n]\!] \xrightarrow{\sigma^j} [\![n-1]\!]$$

has an absolute pushout

$$\begin{tikzcd} {[\![n]\!]} \arrow[r, "\sigma^i"] \arrow[d, "\sigma^j"'] & {[\![n-1]\!]} \arrow[d, "\tau_2"] \\ {[\![n-1]\!]} \arrow[r, "\tau_1"'] & {[\![\ell]\!]} \end{tikzcd}$$

in $\mathcal{Q}_\Sigma$ with both τ_1 and τ_2 in $\mathcal{Q}_\Sigma^-$ and $n-2 \le \ell \le n-1$.

Proof. The functor $i : \mathcal{Q} \to \mathcal{Q}_\Sigma$ preserves absolute pushouts. $\square$

Proof of Lemma 3.24. If $i = j$, then the pushout of (3.8) is $[\![n-1]\!]$ and is preserved by any functor $\mathcal{Q} \to \mathcal{C}$. Suppose $i < j$. Then the square

$$\begin{tikzcd} {[\![n]\!]} \arrow[r, "\sigma^i"] \arrow[d, "\sigma^j"'] & {[\![n-1]\!]} \arrow[d, "\sigma^{j-1}"] \\ {[\![n-1]\!]} \arrow[r, "\sigma^i"'] & {[\![n-2]\!]} \end{tikzcd} \tag{3.9}$$

is a split pushout in $\mathcal{Q}$. Using the notation of Definition 3.22, we define sections

$$d_0 = \delta^{j-1,0}_{n-2} \qquad d_1 = \delta^{j,0}_{n-1} \qquad d'_2 = \delta^{i,0}_{n-1}.$$

Observe that these maps are well-defined, since $1 \le i < j \le n$. Using the cubical relations in Proposition 2.10, we verify the five relations:

$$\begin{array}{ll}
\sigma^j\delta^{j,0} = \mathrm{id}_{[\![n-1]\!]} & (a_2d_1 = \mathrm{id}_C) \\
\sigma^{j-1}\delta^{j-1,0} = \mathrm{id}_{[\![n-2]\!]} & (p_2d_0 = \mathrm{id}_P) \\
\sigma^i\delta^{i,0} = \mathrm{id}_{[\![n-1]\!]} & (a_1d'_2 = \mathrm{id}_B) \\
\delta^{j-1,0}\sigma^i = \sigma^i\delta^{j,0} & (d_0p_1 = a_1d_1) \\
\sigma^j\delta^{i,0}\delta^{j-1,0}\sigma^{j-1} = \delta^{i,0}\sigma^{j-1}\delta^{j-1,0}\sigma^{j-1} = \delta^{i,0}\sigma^{j-1} = \sigma^j\delta^{i,0} & (a_2d'_2d_0p_2 = a_2d'_2).
\end{array}$$

Hence (3.9) is an absolute pushout. $\square$

Lemma 3.27. *Suppose $n \ge 2$. The diagram*

$$[\![n-1]\!] \xleftarrow{\gamma^i} [\![n]\!] \xrightarrow{\gamma^j} [\![n-1]\!] \tag{3.10}$$

has an absolute pushout $[\![\ell]\!]$ in $\mathcal{Q}_\Sigma$ with maps $[\![n-1]\!] \to [\![\ell]\!]$ in $\mathcal{Q}^-_\Sigma$ and $n-2 \le \ell \le n-1$.

Proof. The proof of this lemma is similar to that of Lemma 3.24. Without loss of generality, we may assume that $j \ge i$. We then have three special cases:

(1) ($j = i$) The pushout of (3.10) is $[\![n-1]\!]$ and is absolute.

(2) ($j = i+1$) The square

$$\begin{tikzcd}
[\![n]\!] \arrow[r, "\gamma^i"] \arrow[d, "\gamma^{i+1}"'] & [\![n-1]\!] \arrow[d, "\gamma^i"] \\
[\![n-1]\!] \arrow[r, "\gamma^i"'] & [\![n-2]\!]
\end{tikzcd}$$

is a split pushout: define sections

$$d_0 = \delta^{i+1,1} \qquad d_1 = \delta^{i+2,1} \qquad d'_2 = \delta^{i,1}.$$

Note that $d_1 : [\![n-1]\!] \to [\![n]\!]$ is well-defined, since $i+2$ is at most n. Now we verify the five relations:

$$\begin{array}{ll}
\gamma^{i+1}\delta^{i+2,1} = \mathrm{id}_{[\![n-1]\!]} & (a_2d_1 = \mathrm{id}_C) \\
\gamma^i\delta^{i+1,1} = \mathrm{id}_{[\![n-2]\!]} & (p_2d_0 = \mathrm{id}_P) \\
\gamma^i\delta^{i,1} = \mathrm{id}_{[\![n-1]\!]} & (a_1d'_2 = \mathrm{id}_B) \\
\delta^{i+1,1}\gamma^i = \gamma^i\delta^{i+2,1} & (d_0p_1 = a_1d_1) \\
\gamma^{i+1}\delta^{i,1}\delta^{i+1,1}\gamma^i = \gamma^{i+1}\delta^{i+2,1}\delta^{i,1}\gamma^i = \delta^{i,1}\gamma^i = \gamma^{i+1}\delta^{i,1} & (a_2d'_2d_0p_2 = a_2d'_2).
\end{array}$$

(3) ($j > i+1$) The square

$$\begin{tikzcd}
[\![n]\!] \arrow[r, "\gamma^i"] \arrow[d, "\gamma^{j}"'] & [\![n-1]\!] \arrow[d, "\gamma^{j-1}"] \\
[\![n-1]\!] \arrow[r, "\gamma^i"'] & [\![n-2]\!]
\end{tikzcd}$$

is a split pushout: we define sections

$$d_0 = \delta^{j-1,1} \qquad d_1 = \delta^{j,1} \qquad d'_2 = \delta^{i,1}.$$

The five relations are verified:

$$\begin{array}{ll}
\gamma^j\delta^{j,1} = \mathrm{id}_{[\![n-1]\!]} & (a_2d_1 = \mathrm{id}_C) \\
\gamma^{j-1}\delta^{j-1,1} = \mathrm{id}_{[\![n-2]\!]} & (p_2d_0 = \mathrm{id}_P) \\
\gamma^i\delta^{i,1} = \mathrm{id}_{[\![n-1]\!]} & (a_1d'_2 = \mathrm{id}_B) \\
\delta^{j-1,1}\gamma^i = \gamma^i\delta^{j,1} & (d_0p_1 = a_1d_1) \\
\gamma^j\delta^{i,1}\delta^{j-1,1}\gamma^{j-1} = \gamma^j\delta^{j,1}\delta^{i,1}\gamma^{j-1} = \delta^{i,1}\gamma^{j-1} = \gamma^j\delta^{i,1} & (a_2d'_2d_0p_2 = a_2d'_2).
\end{array}$$ $\square$

Lemma 3.28. *Suppose $n \geq 2$. The diagram*

$$[\![n-1]\!] \xleftarrow{\gamma^i} [\![n]\!] \xrightarrow{\sigma^j} [\![n-1]\!]$$

has an absolute pushout $[\![\ell]\!]$ in $\mathcal{Q}_\Sigma$ with maps $[\![n-1]\!] \to [\![\ell]\!]$ in $\mathcal{Q}_\Sigma^-$ and $n-2 \leq \ell \leq n-1$.

Proof. We have four possibilities:

(1) $(i > j)$ The square

$$\begin{tikzcd} [\![n]\!] \arrow[r, "\sigma^j"] \arrow[d, "\gamma^i"'] & [\![n-1]\!] \arrow[d, "\gamma^{i-1}"] \\ [\![n-1]\!] \arrow[r, "\sigma^j"] & [\![n-2]\!] \end{tikzcd}$$

is a split pushout with sections

$$d_0 = \delta^{i-1,1} \qquad d_1 = \delta^{i,1} \qquad d_2' = \delta^{j,1}.$$

The relations are satisfied:

$$\begin{array}{ll} \gamma^i\delta^{i,1} = \mathrm{id}_{[\![n-1]\!]} & (a_2d_1 = \mathrm{id}_C) \\ \gamma^{i-1}\delta^{i-1} = \mathrm{id}_{[\![n-2]\!]} & (p_2d_0 = \mathrm{id}_P) \\ \sigma^j\delta^{j,1} = \mathrm{id}_{[\![n-1]\!]} & (a_1d_2' = \mathrm{id}_B) \\ \delta^{i-1,1}\sigma^j = \sigma^j\delta^{i,1} & (d_0p_1 = a_1d_1) \\ \gamma^i\delta^{j,1}\delta^{i-1,1}\gamma^{i-1} = \gamma^i\delta^{i,1}\delta^{j,1}\gamma^{i-1} = \delta^{j,1}\gamma^{i-1} = \gamma^i\delta^{j,1} & (a_2d_2'd_0p_2 = a_2d_2'). \end{array}$$

(2) $(i = j)$ The square

$$\begin{tikzcd} [\![n]\!] \arrow[r, "\gamma^i"] \arrow[d, "\sigma^i"'] & [\![n-1]\!] \arrow[d, "\sigma^i"] \\ [\![n-1]\!] \arrow[r, "\sigma^i"] & [\![n-2]\!] \end{tikzcd}$$

is a split pushout with sections

$$d_0 = \delta^{i,0} \qquad d_1 = \delta^{i,0} \qquad d_2' = \delta^{i+1,1}.$$

The relations are satisfied:

$$\begin{array}{ll} \sigma^i\delta^{i,0} = \mathrm{id}_{[\![n-1]\!]} & (a_2d_1 = \mathrm{id}_C) \\ \sigma^i\delta^{i,0} = \mathrm{id}_{[\![n-2]\!]} & (p_2d_0 = \mathrm{id}_P) \\ \gamma^i\delta^{i+1,1} = \mathrm{id}_{[\![n-1]\!]} & (a_1d_2' = \mathrm{id}_B) \\ \delta^{i,0}\sigma^i = \gamma^i\delta^{i,0} & (d_0p_1 = a_1d_1) \\ \sigma^i\delta^{i+1,1}\delta^{i,0}\sigma^i = \sigma^i\delta^{i,0}\delta^{i,1}\sigma^i = \delta^{i,1}\sigma^i = \sigma^i\delta^{i+1,1} & (a_2d_2'd_0p_2 = a_2d_2'). \end{array}$$

(3) $(i + 1 = j)$ The square

$$\begin{tikzcd} [\![n]\!] \arrow[r, "\gamma^i"] \arrow[d, "\sigma^j"'] & [\![n-1]\!] \arrow[d, "\sigma^i"] \\ [\![n-1]\!] \arrow[r, "\sigma^i"] & [\![n-2]\!] \end{tikzcd}$$

is a split pushout with sections

$$d_0 = \delta^{i,0} \qquad d_1 = \delta^{j,0} \qquad d_2' = \delta^{i,1}.$$

The relations are satisfied:

$$\begin{array}{ll} \sigma^j\delta^{j,0} = \mathrm{id}_{[\![n-1]\!]} & (a_2d_1 = \mathrm{id}_C) \\ \sigma^i\delta^{i,0} = \mathrm{id}_{[\![n-2]\!]} & (p_2d_0 = \mathrm{id}_P) \\ \gamma^i\delta^{i,1} = \mathrm{id}_{[\![n-1]\!]} & (a_1d_2' = \mathrm{id}_B) \\ \delta^{i,0}\sigma^i = \gamma^i\delta^{j,0} & (d_0p_1 = a_1d_1) \\ \sigma^j\delta^{i,1}\delta^{i,0}\sigma^i = \sigma^j\delta^{j,0}\delta^{i,1}\sigma^i = \delta^{i,1}\sigma^i = \sigma^j\delta^{i,1} & (a_2d_2'd_0p_2 = a_2d_2'). \end{array}$$

(4) $(i+1<j)$ The square

$$\begin{tikzcd} \llbracket n \rrbracket \arrow[r, "\gamma^i"] \arrow[d, "\sigma^j"'] & \llbracket n-1 \rrbracket \arrow[d, "\sigma^{j-1}"] \\ \llbracket n-1 \rrbracket \arrow[r, "\gamma^i"] & \llbracket n-2 \rrbracket \end{tikzcd}$$

is a split pushout with sections

$$d_0 = \delta^{j-1,0} \qquad d_1 = \delta^{j,0} \qquad d_2' = \delta^{i,1}.$$

The relations are satisfied:

$$\begin{array}{ll} \sigma^j \delta^{j,0} = \mathrm{id}_{\llbracket n-1 \rrbracket} & (a_2 d_1 = \mathrm{id}_C) \\ \sigma^{j-1}\delta^{j-1,0} = \mathrm{id}_{\llbracket n-2 \rrbracket} & (p_2 d_0 = \mathrm{id}_P) \\ \gamma^i \delta^{i,1} = \mathrm{id}_{\llbracket n-1 \rrbracket} & (a_1 d_2' = \mathrm{id}_B) \\ \delta^{j-1,0}\gamma^i = \gamma^i \delta^{j,0} & (d_0 p_1 = a_1 d_1) \\ \sigma^j \delta^{i,1}\delta^{j-1,0}\sigma^{j-1} = \sigma^j\delta^{j,0}\delta^{i,1}\sigma^{j-1} = \delta^{i,1}\sigma^{j-1} = \sigma^j\delta^{i,1} & (a_2 d_2' d_0 p_2 = a_2 d_2'). \quad \square \end{array}$$

Corollary 3.29. *The category $\mathcal{Q}_\Sigma$ satisfies axiom* EZ3 *of Definition* 3.2.

4. The symmetric cubical site models the homotopy category

In this section, we equip q_Σ**Set** with a model structure Quillen equivalent to s**Set**. This is the heart of the paper. We start by describing a spatial model structure on q**Set**. We then lift the model structure from q**Set** along the restriction functor $i^* : q_\Sigma\mathbf{Set} \to q\mathbf{Set}$. In order to do this, we need to check that cell complexes in q_Σ**Set** built out of the representable functors are well-behaved homotopically. The outline of the argument is standard; as usual, it requires some work to verify. The resulting Quillen pair $i_! \dashv i^*$ is readily shown to be a Quillen equivalence. Finally, we discuss the monoidal properties of the lifted model structure on q_Σ**Set**.

4.1. The homotopy theory of cubical sets

In [9], Cisinski proves that $\mathcal{Q}$ is a test category and thus the category $q\mathbf{Set} = \mathbf{Set}^{\mathcal{Q}^{\mathrm{op}}}$ models spaces. Jardine gives a survey in English of Cisinski's methods in [23]. We will summarize Cisinski's results here. Recall that $\partial\Box^n$ is the subpresheaf of $\Box^n$ given by

$$(\partial\Box^n)_m = \left\{ f : \llbracket m \rrbracket \to \llbracket n \rrbracket \in \mathcal{Q} \;\middle|\; f \text{ factors as } f : \llbracket m \rrbracket \to \llbracket k \rrbracket \to \llbracket n \rrbracket, k < n \right\}.$$

This comes equipped with a monomorphism $\partial\Box^n \to \Box^n$. Put another way, $\partial\Box^n$ is the union of the $(n-1)$-dimensional faces of $\Box^n$. We define the i, ε*-cap*

$$(\sqcap^n_{i,\varepsilon})_m = \left\{ f : \llbracket m \rrbracket \to \llbracket n \rrbracket \in \mathcal{Q} \;\middle|\; f \text{ factors as } f : \llbracket m \rrbracket \to \llbracket n-1 \rrbracket \xrightarrow{d} \llbracket n \rrbracket, d \neq \delta_n^{\varepsilon,i} \right\}$$

for $1 \leq i \leq n$. This comes equipped with a monomorphism $\sqcap^n_{i,\varepsilon} \to \partial\Box^n$.

Definition 4.1. We say a functor $F : \mathcal{B} \to \mathcal{C}$ of small categories is a *Thomason equivalence* if F induces a weak equivalence $NF : N\mathcal{B} \to N\mathcal{C}$ on nerves. Let $\mathcal{A}$ be a small category. We say a map $f : X \to Y$ in $\widehat{\mathcal{A}}$ is an ∞*-equivalence* if f induces a Thomason equivalence

$$\mathcal{A} \downarrow f : \mathcal{A} \downarrow X \to \mathcal{A} \downarrow Y$$

of categories.

Definition 4.2. The *simplicial realization* of a cubical set $X \in q\mathbf{Set}$ is the colimit

$$|X| = \operatorname*{colim}_{\Box^n \to X} (\Delta[1])^n$$

of simplicial sets.

Note that simplicial realization is the unique cocontinuous functor $q\mathbf{Set} \to s\mathbf{Set}$ taking $\Box^n$ to $(\Delta[1])^n$. Since its restriction to $\mathcal{Q}$ is strong monoidal, it is strong monoidal on q**Set**. We can now state the following theorem:

Theorem 4.3 (*[9, Théorème 8.4.38]*)**.** (1) *The category q**Set** forms a proper model category with cofibrations monomorphisms and weak equivalences the ∞-equivalences. We call this model structure the* spatial model structure. *It is cofibrantly generated with generating cofibrations*

$$\{\partial\Box^n \to \Box^n \mid n \geq 0\}$$

and generating acyclic cofibrations

$$\{\sqcap^n_{i,\varepsilon} \to \square^n \mid 1 \le i \le n, \varepsilon = 0, 1\}.$$

(2) *The spatial model structure is monoidal: if $i : A \to B$ and $j : K \to L$ are cofibrations,*

$$i \odot j : A \otimes L \coprod_{A \otimes K} B \otimes K \to B \otimes L$$

is a cofibration, acyclic if either i or j is.

(3) *Simplicial realization is a left Quillen equivalence $q\mathbf{Set} \to s\mathbf{Set}$.*

Theorem 4.3 is the basis of everything that follows. We will take it for granted. Jardine also gives a proof of it in the survey [23] following Cisinski's methods.

4.2. Homotopy and asphericity

Recall that the categories $\mathcal{Q}$ and $\mathcal{Q}_\Sigma$ are related by an inclusion functor $i : \mathcal{Q} \to \mathcal{Q}_\Sigma$. This produces an adjoint pair

$$i_! : q\mathbf{Set} \rightleftarrows q_\Sigma\mathbf{Set} : i^*$$

given by left Kan extension and restriction.

Proposition 4.4. *The functors $i_!$ and i^* are strong and lax monoidal, respectively.*

Proof. That $i_!$ is strong monoidal is a consequence of Proposition A.1: since the square

$$\begin{tikzcd} \mathcal{Q} \arrow[r, "i"] \arrow[d, "{[-]}"'] \arrow[dr, phantom, "\Downarrow"] & \mathcal{Q}_\Sigma \arrow[d, "{[-]}"] \\ q\mathbf{Set} \arrow[r, "i_!"'] & q_\Sigma\mathbf{Set} \end{tikzcd}$$

commutes up to natural isomorphism and i is strong monoidal, the extension $i_!$ is strong monoidal. Now suppose K and L are extended cubical sets. The counit of the adjunction $i_! \dashv i^*$ together with the monoidalness of $i_!$ yields a natural map

$$i_!(i^*K \otimes i^*L) \xrightarrow{\sim} i_!i^*K \otimes i_!i^*L \longrightarrow K \otimes L\ .$$

The adjoint is a natural transformation

$$i^*K \otimes i^*L \to i^*(K \otimes L)$$

making i^* lax monoidal. $\square$

Definition 4.5. Suppose $n > 0$. Let $\{\varepsilon\}$ denote the formal $(0, n)$-product

$$\underbrace{(\varepsilon, \ldots, \varepsilon)}_{n \text{ entries}}$$

for $\varepsilon = 0, 1$. Note that $\{\varepsilon\} = (d^{1,\varepsilon})^n$. Suppose $f, g : X \to Y$ are two maps in $q_\Sigma\mathbf{Set}$. We say f and g are *$\square^n_\Sigma$-homotopic* if there is a filler h in the diagram

$$\begin{tikzcd} X \otimes \square^0_\Sigma \arrow[d, "\mathrm{id}\otimes\{0\}"'] \arrow[dr, "f"] & \\ X \otimes \square^n_\Sigma \arrow[r, dotted, "h"] & Y. \\ X \otimes \square^0_\Sigma \arrow[u, "\mathrm{id}\otimes\{1\}"] \arrow[ur, "g"'] & \end{tikzcd}$$

We call h a *$\square^n_\Sigma$-homotopy* from f to g. By abuse of terminology, we will sometimes simply call f and g *homotopic*. We say a map k is a *homotopy equivalence* if there is a map ℓ so that $k\ell$ and ℓk are both homotopic to the identity. We define $\square^n$-homotopy in $q\mathbf{Set}$ similarly.

Note that $\square^n_\Sigma$-homotopy is not an equivalence relation for arbitrary Y. The extended cubical set Y must possess a sort of homotopy extension property: Y must be fibrant. This is precisely the same reason that $\Delta[1]$-homotopy is not an equivalence relation on maps in $s\mathbf{Set}$ unless the maps have a Kan complex as their target. Using the spatial model structure on $q\mathbf{Set}$, we have the following standard result:

Proposition 4.6. *Suppose $k : X \to Y$ is a homotopy equivalence in $q\mathbf{Set}$. Then k is an ∞-equivalence.*

Lemma 4.7. *Suppose f and $g : X \to Y$ are $\square^n_\Sigma$-homotopic maps in q_Σ**Set**. Then i^*f and i^*g are $\square^n$-homotopic.*

Proof. Let $h : X \otimes \square^n_\Sigma \to Y$ be a homotopy from f to g. By Proposition 4.4, i^* is lax monoidal, so we have a diagram

$$\begin{tikzcd}
i^*X \otimes \square^0 \arrow[r, "\sim"] \arrow[d, "\mathrm{id}\otimes\{0\}"'] & i^*X \otimes i^*\square^0_\Sigma \arrow[r, "\sim"] \arrow[d, "\mathrm{id}\otimes i^*\{0\}"'] & i^*(X \otimes \square^0_\Sigma) \arrow[d, "i^*(\mathrm{id}\otimes\{0\})"'] \arrow[dr, "i^*f"] & \\
i^*X \otimes \square^n \arrow[r] & i^*X \otimes i^*\square^n_\Sigma \arrow[r] & i^*(X \otimes \square^n_\Sigma) \arrow[r, "i^*h"] & i^*Y. \\
i^*X \otimes \square^0 \arrow[r, "\sim"] \arrow[u, "\mathrm{id}\otimes\{1\}"] & i^*X \otimes i^*\square^0_\Sigma \arrow[r, "\sim"] \arrow[u, "\mathrm{id}\otimes i^*\{1\}"] & i^*(X \otimes \square^0_\Sigma) \arrow[u, "i^*(\mathrm{id}\otimes\{1\})"] \arrow[ur, "i^*g"'] &
\end{tikzcd}$$

The unit $\square^0 \to i^*\square^0_\Sigma$ is an isomorphism since $\square^0_\Sigma$ and $\square^0$ are terminal and i^* is a right adjoint. Hence the top and bottom horizontal arrows are isomorphisms. As a result, the horizontal chain of arrows is a $\square^n$-homotopy between i^*f and i^*g. $\square$

Corollary 4.8. *The functor i^* preserves homotopy equivalences.*

Lemma 4.9. *The inclusion functor $i : \mathcal{Q} \to \mathcal{Q}_\Sigma$ is aspherical, i.e., for all $n \geq 0$,*

$$i \downarrow [\![n]\!] \to \mathcal{Q}_\Sigma \downarrow [\![n]\!]$$

is a Thomason equivalence.

Proof. We define a map $H : [\![2n]\!] \to [\![n]\!]$ as the formal product

$$(x_1 \wedge x_{n+1}, x_2 \wedge x_{n+2}, \ldots, x_n \wedge x_{2n}).$$

This is an application of a symmetry followed by n connections. The map H gives a homotopy between $\{0\}$ and the identity map on $\square^n_\Sigma$:

$$\begin{tikzcd}
\square^n_\Sigma \arrow[d, "\mathrm{id}\otimes\{0\}"'] \arrow[dr, "\{0\}"] & \\
\square^n_\Sigma \otimes \square^n_\Sigma \arrow[r, "H"] & \square^n_\Sigma. \\
\square^n_\Sigma \arrow[u, "\mathrm{id}\otimes\{1\}"] \arrow[ur, "\mathrm{id}"'] &
\end{tikzcd}$$

Hence the inclusion $\{0\}$ is a homotopy equivalence in q_Σ**Set**. By Corollary 4.8,

$$i^*\{0\} : \square^0 \to i^*\square^n_\Sigma$$

is a homotopy equivalence and hence ∞-equivalence in q**Set**. Thus

$$\mathcal{Q} \to \mathcal{Q} \downarrow i^*\square^n_\Sigma$$

is a Thomason equivalence and $\mathrm{N}(i^*\square^n_\Sigma)$ is contractible. Since $\mathcal{Q} \downarrow i^*\square^n_\Sigma$ is equivalent to $i \downarrow \square^n_\Sigma$, we conclude that

$$i \downarrow [\![n]\!] \to \mathcal{Q}_\Sigma \downarrow [\![n]\!]$$

is a Thomason equivalence. $\square$

Proposition 4.10. *Suppose $X \in q_\Sigma$**Set**. Then the functor*

$$\mathcal{Q} \downarrow i^*X \to \mathcal{Q}_\Sigma \downarrow X$$

induced by i induces an equivalence of nerves.

Proof. This is a special case of [27, Proposition 1.2.9]. Suppose $s : \square^n_\Sigma \to X$ is a cube of X. Consider the category

$$(\mathcal{Q} \downarrow i^*X) \downarrow s :$$

this is the category of triangles

$$\begin{tikzcd}
\square^m_\Sigma \arrow[r] \arrow[d] & X \\
\square^n_\Sigma \arrow[ur, "s"'] &
\end{tikzcd}$$

with morphisms diagrams of the shape

$$\begin{tikzcd} \square_\Sigma^{m'} \arrow[r, "i(f)"] \arrow[dr] & \square_\Sigma^{m} \arrow[r] \arrow[d] & X \\ & \square_\Sigma^{n} \arrow[ur, "s"'] & \end{tikzcd}$$

The functor

$$(\mathscr{Q} \downarrow i^*X) \downarrow s \to \mathscr{Q} \downarrow \square_\Sigma^n$$

forgetting the map to X has a left adjoint given by composition with s, so it is a Thomason equivalence. By Lemma 4.9, we may conclude that

$$(\mathscr{Q} \downarrow i^*X) \downarrow s$$

has contractible nerve, so by Quillen's Theorem A [32],

$$\mathscr{Q} \downarrow i^*X \to \mathscr{Q}_\Sigma \downarrow X$$

is a Thomason equivalence. $\square$

Corollary 4.11. *The functor i^* reflects ∞-equivalences; i.e., $X \to Y$ in q_Σ**Set** induces a Thomason equivalence*

$$\mathscr{Q}_\Sigma \downarrow X \to \mathscr{Q}_\Sigma \downarrow Y$$

if and only if

$$\mathscr{Q} \downarrow i^*X \to \mathscr{Q} \downarrow i^*Y$$

is a Thomason equivalence.

4.3. A Quillen equivalence

We show that $i_! \dashv i^*$ is a Quillen equivalence simultaneously with the construction of the spatial model structure on q_Σ**Set**.

Proposition 4.12. *The functor $i_! : q\mathbf{Set} \to q_\Sigma\mathbf{Set}$ preserves monomorphisms.*

Before we embark on this, recall that $\partial\square_\Sigma^n$ is the subpresheaf of $\square_\Sigma^n$ given by

$$\partial\square_\Sigma^n([\![m]\!]) = \left\{ f \in \mathscr{Q}_\Sigma([\![m]\!], [\![n]\!]) \;\middle|\; f \text{ factors } [\![m]\!] \to [\![k]\!] \to [\![n]\!], k < n \text{ in } \mathscr{Q}_\Sigma \right\}.$$

Another description of $\partial\square_\Sigma^n([\![m]\!])$ is as the set of formal (m, n)-products with at least one entry 0 or 1. As we will describe below, $\partial\square_\Sigma^n$ is the union of the faces of $\square_\Sigma^n$.

Proof of Proposition 4.12. By Corollary 3.20, the map $i_!\mathrm{sk}_{n-1}\square^n \to \mathrm{sk}_{n-1}\square_\Sigma^n$ is an isomorphism. This implies that $i_!(\partial\square^n \to \square^n)$ is, up to isomorphism, the map $\partial\square_\Sigma^n \to \square_\Sigma^n$. Since

$$\mathsf{mono} = \mathsf{Cell}\left\{ \partial\square^n \to \square^n \;\middle|\; n \geq 0 \right\},$$

we may conclude that $i_!$ preserves all monomorphisms in q**Set**. $\square$

Lemma 4.13. *Suppose Y is an n-skeletal cubical set. The map*

$$Y \coprod_{\mathrm{sk}_{n-1}Y} i^*i_!\mathrm{sk}_{n-1}Y \to i^*i_!Y$$

is a monomorphism.

Proof. First note that the corner map

$$i^*\partial\square_\Sigma^n \coprod_{\partial\square^n} \square^n \to i^*\square_\Sigma^n$$

is a monomorphism. This is a consequence of the fact that i is faithful: for $[\![m]\!]$, we have

$$\left(i^*\partial\square_\Sigma^n \coprod_{\partial\square^n} \square^n \right)_m = \left\{ f : [\![m]\!] \to [\![n]\!] \;\middle|\; f \text{ factors through } [\![k]\!], k < n, \text{or } f \in \mathrm{ar}\mathscr{Q} \right\}.$$

Let S be the set of nondegenerate n-simplices of Y. By Propositions 3.10 and 3.4, we may write Y as a pushout

$$\begin{tikzcd} \coprod_S \partial\square^n \arrow[r] \arrow[d] & \mathrm{sk}_{n-1}Y \arrow[d] \\ \coprod_S \square^n \arrow[r] & Y \end{tikzcd}$$

where S is the set of nondegenerate n-simplices of Y. Write $\eta : \mathrm{id} \to i^*i_!$ for the unit of the adjunction $i_! \dashv i^*$. Consider the cube

$$\begin{tikzcd}[row sep=small, column sep=small] & \coprod_S i^*i_!\partial\square^n \arrow[rr] \arrow[dd] & & i^*i_!\mathrm{sk}_{n-1}Y \arrow[dd] \\ \coprod_S \partial\square^n \arrow[ur, "\coprod_S \eta_{\partial\square^n}"] \arrow[rr, crossing over] \arrow[dd, "j"'] & & \mathrm{sk}_{n-1}Y \arrow[ur, "\eta_{\mathrm{sk}_{n-1}Y}"'] & \\ & \coprod_S i^*i_!\square^n \arrow[rr] & & i^*i_!Y. \\ \coprod_S \square^n \arrow[ur, "\coprod_S \eta_{\square^n}"] \arrow[rr] & & Y \arrow[ur, "\eta_Y"'] \arrow[from=uu, crossing over] & \end{tikzcd} \tag{4.1}$$

The functors i^* and $i_!$ both preserve colimits, so the front and back faces are both pushouts. As a result, the square

$$\begin{tikzcd} \coprod_S \left(i^*\partial\square^n_\Sigma \coprod_{\partial\square^n} \square^n\right) \arrow[r] \arrow[d] & i^*i_!\mathrm{sk}_{n-1}Y \coprod_{\mathrm{sk}_{n-1}Y} Y \arrow[d, "g"] \\ \coprod_S i^*i_!\square^n \arrow[r] & i^*i_!Y \end{tikzcd}$$

is a pushout; since $q\mathbf{Set}$ is a topos, the arrow g is a monomorphism. □

Lemma 4.14. *An arbitrary small coproduct of ∞-equivalences in $q\mathbf{Set}$ is an ∞-equivalence.*

Proof. This is a standard model category result (Ken Brown's lemma [20], together with the fact that everything in $q\mathbf{Set}$ is cofibrant). Alternatively, observe that $|-|$ reflects weak equivalences and preserves small coproducts; arbitrary small coproducts of weak equivalences in $s\mathbf{Set}$ are themselves weak equivalences. □

Proposition 4.15. *The unit $\eta : \mathrm{id} \to i^*i_!$ of the adjunction $i_! \dashv i^*$ is a natural ∞-equivalence in $q\mathbf{Set}$.*

Proof. We will first prove that η_X is an ∞-equivalence for skeletal X by induction on the dimension. If X is 0-skeletal, then $X = \coprod_S \square^0$ and η_X is an isomorphism. Let $n > 0$ and suppose η_X is a weak equivalence for all $(n-1)$-skeletal X. In particular, $\eta_{\partial\square^n}$ is an ∞-equivalence since $\partial\square^n$ is the $(n-1)$-skeleton of $\square^n$. Suppose Y is n-skeletal. From Corollary 4.11, we know that $\square^n \to i^*\square^n_\Sigma$ is an ∞-equivalence. Recall the cube (4.1) in the proof of Lemma 4.13. The front and back faces are both pushout squares. The arrows j and $i^*i_!j$ are both monomorphisms by Proposition 4.12. Every object of $q\mathbf{Set}$ is cofibrant, so these pushout squares are both homotopy cocartesian. By Lemma 4.14 and our assumptions, the diagonal arrows $\coprod_S \eta_{\square^n}$, $\coprod_S \eta_{\partial\square^n}$ and $\eta_{\mathrm{sk}_{n-1}Y}$ are weak equivalences, so η_Y is an ∞-equivalence. By induction, we may conclude that η_X is an ∞-equivalence for all X which are n-skeletal for some n.

Suppose X is an arbitary cubical set. Now consider the ladder

$$\begin{tikzcd} \mathrm{sk}_0X \arrow[r] \arrow[d, "\eta_0"] & \mathrm{sk}_1X \arrow[r] \arrow[d, "\eta_1"] & \cdots \arrow[r] & \mathrm{sk}_nX \arrow[r] \arrow[d, "\eta_n"] & \cdots \\ i^*i_!\mathrm{sk}_0X \arrow[r] & i^*i_!\mathrm{sk}_1X \arrow[r] & \cdots \arrow[r] & i^*i_!\mathrm{sk}_nX \arrow[r] & \cdots \end{tikzcd}$$

By Lemma 4.13, this map is an acyclic Reedy cofibration: the map η_0 is a cofibration, the corner maps

$$i^*i_!\mathrm{sk}_{n-1}X \coprod_{\mathrm{sk}_{n-1}X} \mathrm{sk}_n X \to i^*i_!\mathrm{sk}_n X$$

are cofibrations, and each η_i is an ∞-equivalence. Hence the colimit $\eta_X : X \to i^*i_!X$ is an ∞-equivalence. $\square$

Corollary 4.16. *The counit $\varepsilon : i_!i^* \to \mathrm{id}$ of the adjunction $i_! \dashv i^*$ is a natural ∞-equivalence in $\mathcal{Q}_\Sigma$.*

Proof. Suppose $X \in q_\Sigma\mathbf{Set}$. Consider the triangle

$$\begin{tikzcd} i^*X \arrow[r, "\eta_{i^*X}"] \arrow[dr, "\mathrm{id}"'] & i^*i_!i^*X \arrow[d, "i^*\varepsilon_X"] \\ & i^*X. \end{tikzcd}$$

The map η_{i^*X} is an ∞-equivalence by Proposition 4.15, so $i^*\varepsilon_X$ is an ∞-equivalence. By Proposition 4.10, ε_X is an ∞-equivalence. $\square$

We can finally prove the main theorem of this paper.

Theorem 4.17. *The category $q_\Sigma\mathbf{Set}$ forms a left proper cofibrantly generated model category known as the* spatial model structure *with weak equivalences the ∞-equivalences and fibrations the maps $p : X \to Y$ which are fibrations in $q\mathbf{Set}$ upon application of the restriction i^*. The set of generating cofibrations is*

$$I = \{\partial\square^n_\Sigma \to \square^n_\Sigma \mid n \geq 0\}$$

and the set of generating acyclic cofibrations is

$$J = \{i_!\sqcap^n_{j,\varepsilon} \to \square^n_\Sigma \mid 1 \leq j \leq n \text{ and } \varepsilon = 0, 1\}.$$

Proof. This is a consequence of a standard result on lifting model structures along an adjunction—see, for example, [34]. The key point is the following: suppose

$$\begin{tikzcd} i_!\sqcap^n_{j,\varepsilon} \arrow[r] \arrow[d, "i_!e"'] & A \arrow[d, "f"] \\ i_!\square^n \arrow[r] & B \end{tikzcd}$$

is a pushout in $q_\Sigma\mathbf{Set}$. The functor i^* preserves all colimits and limits, so the right square in

$$\begin{tikzcd} \sqcap^n_{j,\varepsilon} \arrow[r] \arrow[d, "e"'] & i^*i_!\sqcap^n_{j,\varepsilon} \arrow[r] \arrow[d, "i^*i_!e"'] & i^*A \arrow[d, "i^*f"] \\ \square^n \arrow[r] & i^*i_!\square^n \arrow[r] & i^*B \end{tikzcd}$$

is a pushout in $q\mathbf{Set}$. But by Proposition 4.15, $i^*i_!e$ is a weak equivalence. By Proposition 4.12, $i^*i_!e$ is a monomorphism. Hence i^*f is an ∞-equivalence in $q\mathbf{Set}$, so by Corollary 4.11 f is an ∞-equivalence. Since $q_\Sigma\mathbf{Set}$ is locally presentable, we can use the small object argument to factor every arrow in $q_\Sigma\mathbf{Set}$ as a map in Cell$J$ followed by a J-injective map [1]. But by the above discussion—together with the fact that i^* preserves filtered colimits—the maps in Cell$J$ are acyclic cofibrations and the maps in Inj$J$ are fibrations. For left properness, apply the functor i^* to the necessary diagram and note that i^* preserves cofibrations. $\square$

Since $\Delta[1]$ is a cubical monoid, we may define the *extended geometric realization* functor $|-|_\Sigma$ to be the unique cocontinuous strong monoidal functor $\mathcal{Q}_\Sigma \to s\mathbf{Set}$ taking $\square^1_\Sigma$ to $\Delta[1]$.

Theorem 4.18. *The functors $i_!$ and $|-|_\Sigma$ are both left Quillen equivalences. The diagram*

$$\begin{tikzcd} q\mathbf{Set} \arrow[rr, "i_!"] \arrow[dr, "|-|"'] & & q_\Sigma\mathbf{Set} \arrow[dl, "|-|_\Sigma"] \\ & s\mathbf{Set} & \end{tikzcd} \tag{4.2}$$

commutes up to natural isomorphism.

Proof. We have proved that the unit and counit $\eta : \mathrm{id} \to i^*i_!$ and $\varepsilon : i_!i^* \to \mathrm{id}$ are natural ∞-equivalences (4.15 and Corollary 4.16). Strictly speaking, this is not the right condition for Quillen equivalences, as we would need to use the derived left and right adjoints. However, i^* and $i_!$ coincide with their derived functors: $i_!$ is left Quillen, but everything in q**Set** is cofibrant, so it preserves all ∞-equivalences. The right adjoint i^* preserves all ∞-equivalences as well (Proposition 4.10). Two-out-of-three ensures that $|-|_\Sigma$ is a left Quillen equivalence. $\square$

Remark 4.19. A few notes about Theorem 4.17 are in order. Not all monomorphisms in the spatial model structure on q_Σ**Set** are cofibrations and not all objects in the spatial model structure are cofibrant. For example, the quotient by Σ_n of $\partial\square^n_\Sigma \to \square^n_\Sigma$ is a monomorphism, but if $n > 1$, it is not a cofibration. In fact, the quotient of $\square^n_\Sigma$ by Σ^n is not cofibrant if $n > 1$. As a result $|-|_\Sigma$ may not reflect weak equivalences. However, its left derived functor $\mathbf{L}|-|_\Sigma$ preserves and reflects weak equivalences.

We have shown that i^* is a left and right Quillen functor. On the level of homotopy categories, since $i_! \dashv i^*$ induces an equivalence of Hoq**Set** with Hoq_Σ**Set**, the adjoint pair $i^* \dashv \mathbf{R}i_*$ must also induce an equivalence of Hoq**Set** with Hoq_Σ**Set**, and so $\mathbf{R}i_*$ and $i_!$ coincide.

4.4. The extended product

We have now shown that q_Σ**Set** models spaces. In the remainder of this section, we will prove that the monoidal structure on q_Σ**Set** is compatible with the spatial model structure.

Lemma 4.20. *Suppose X is an extended cubical set and $n \geq 0$. The map $\pi : X \otimes \square^n_\Sigma \to X$ given by the product of the identity map on X and the unique map $\square^n_\Sigma \to \square^0_\Sigma$ is an ∞-equivalence.*

Proof. This is essentially the same as the proof of Lemma 4.9. It is sufficient to prove for all X when $n = 1$. Let $IX = X \otimes \square^1_\Sigma$ and let s be the map

$$s = \mathrm{id}_X \otimes \{0\} : X \to IX.$$

Then $\pi s = \mathrm{id}_X$. We have a homotopy

$$\begin{tikzcd}
X \otimes \square^1_\Sigma \arrow[d, "\mathrm{id}_{IX}\otimes\{0\}"'] \arrow[dr, "s\pi"] & \\
X \otimes \square^2_\Sigma \arrow[r, "\mathrm{id}_X\otimes\gamma^1"] & X \otimes \square^1_\Sigma \\
X \otimes \square^1_\Sigma \arrow[u, "\mathrm{id}_{IX}\otimes\{1\}"] \arrow[ur, "\mathrm{id}_{IX}"'] &
\end{tikzcd}$$

between id_{IX} and $s\pi$, so π is a homotopy equivalence. By Corollary 4.8, $i^*\pi$ is a homotopy equivalence and hence ∞-equivalence, so π is an ∞-equivalence. $\square$

Lemma 4.21. *Suppose $i : A \to B$ and $j : K \to L$ are monomorphisms in q_Σ**Set**. Then the pushout-product*

$$i \odot j : A \otimes L \coprod_{A\otimes K} B \otimes K \to B \otimes L$$

is a monomorphism.

Proof. Recall from Proposition 3.16 that

$$I_\Sigma = \left\{ (\partial\square^n_\Sigma)_H \to (\square^n_\Sigma)_H \;\middle|\; n \geq 0 \text{ and } H \leqslant \Sigma_n \right\}$$

is a cellular model for q_Σ**Set**. First, we will show that the pushout-product of any two maps in I_Σ is a monomorphism. Suppose $n, m \geq 0$ and H_n, H_m subgroups of Σ_n and Σ_m, respectively. Let $H = H_n \times H_m$ be the subgroup of Σ_{n+m} generated by the images of H_n and H_m under the homomorphism $\Sigma_n \times \Sigma_m \to \Sigma_{n+m}$. Then

$$(\partial\square^n_\Sigma \to \square^n_\Sigma)_{H_n} \odot (\partial\square^m_\Sigma \to \square^m_\Sigma)_{H_m} \cong (\partial\square^{n+m}_\Sigma \to \square^{n+m}_\Sigma)_H.$$

This map is a monomorphism. A standard deduction lets us upgrade this to deduce that the pushout-product of any two monomorphisms is a monomorphism: by the small object argument applied to I_Σ, we know that j is a monomorphism if and only if $j \pitchfork p$ for all $p \in \mathrm{Inj}I_\Sigma$ [1]. $\square$

Theorem 4.22. *The spatial model structure on q_Σ**Set** is monoidal and satisfies the Schwede–Shipley monoid axiom.*

Proof. That the spatial model structure is monoidal is a straightforward consequence of the fact that the generating cofibrations and acyclic cofibrations are given by left Kan extension along i, which is itself strong monoidal. That is, if f and g are cofibrations in q**Set**, then $i_!f \odot i_!g \cong i_!(f \odot g)$ is a cofibration in q_Σ**Set**, acyclic if either f or g is.

For the monoid axiom, first note that it is sufficient to check that

$$\mathrm{Cell}\left\{ X \otimes \sqcap^n_{i,\varepsilon} \to X \otimes \square^n \;\middle|\; X \in \mathrm{ob}\,q_\Sigma\mathbf{Set},\ 1 \leq i \leq n \text{ and } \varepsilon = 0, 1 \right\}$$

comprises ∞-equivalences by [34, Lemma 3.5 (2)]. Let $\varepsilon = 0$ or 1 and let $n > 0$. For arbitrary extended cubical sets Y, the map

$$\mathrm{id}_Y \otimes \{\varepsilon\} : Y \otimes \square^0_\Sigma \to Y \otimes \square^1_\Sigma$$

is a section of an ∞-equivalence by Lemma 4.20, so it is itself a weak equivalence. Now consider the pushout

$$\begin{tikzcd}
X \otimes \partial\square^{n-1}_\Sigma \arrow[r, "\mathrm{id}\otimes\{1-\varepsilon\}"] \arrow[d, "g"'] & X \otimes \partial\square^{n-1}_\Sigma \otimes \square^1_\Sigma \arrow[d] \arrow[ddr, bend left] & \\
X \otimes \square^{n-1}_\Sigma \arrow[r, "k"] \arrow[drr, bend right, "\mathrm{id}\otimes\{1-\varepsilon\}"'] & X \otimes i_! \sqcap^n_{n,\varepsilon} \arrow[dr, "\ell"] & \\
 & & X \otimes \square^n_\Sigma .
\end{tikzcd}$$

By the two-out-of-three axiom and Lemma 4.20 we know that ℓ is an ∞-equivalence if and only if k is an ∞-equivalence. But g is a monomorphism by Lemma 4.21. so i^*g is a monomorphism. Since i^*k is the cobase change of an ∞-equivalence along a cofibration and $q\mathbf{Set}$ is left proper, i^*k is an ∞-equivalence, so k is an ∞-equivalence. The cosymmetry maps allow us to permute the lower cap coordinate n. $\square$

5. Diagrams of extended cubical sets and regularity

Recall that if X is a simplicial set, there is a weak equivalence

$$\operatorname*{hocolim}_{\Delta[n]\to X} \Delta[n] \to X$$

induced by the identification of X with the colimit of its simplices. There are various ways to prove this. One method uses Reedy model structures to show that the honest colimit of the diagram of simplices of X computes the homotopy colimit. In this section, we will prove an analogous formula for (extended) cubical sets: these can be decomposed as the homotopy colimit of their cubes.

Suppose $\mathscr{R}$ is a Reedy category and $\mathscr{C}$ is a model category. The category $\mathscr{C}^{\mathscr{R}}$ of $\mathscr{R}$-diagrams in $\mathscr{C}$ may be equipped with *Reedy model structure* [20,19,12,33]. This by now is a well-known construction; we have implicitly used it in describing directed colimits and pushouts of weak equivalences. To fix notation, we recall the definitions here:

Definition 5.1 (*[19, Chapter 15]*)**.** Suppose $r \in \mathrm{ob}\mathscr{R}$.

(1) We define $\partial(\mathscr{R}^+ \downarrow r)$ to be the full subcategory of $\mathscr{R}^+ \downarrow r$ consisting of non-identity arrows $s \to r$. Let $F \in \mathscr{C}^{\mathscr{R}}$. The *rth latching object of F* is the colimit

$$L_rF = \operatorname*{colim}_{\partial(\mathscr{R}^+\downarrow r)} F \in \mathscr{C}.$$

This is functorial in F. Note there is a natural map $L_rF \to F_r$. Suppose $f : F \to G$ is an arrow in $\mathscr{C}^{\mathscr{R}}$. We say f is a *Reedy cofibration* if each corner map

$$F_r \coprod_{L_r(F)} L_r(G) \to G_r,$$

$r \in \mathrm{ob}\mathscr{R}$, is a cofibration in $\mathscr{C}$.

(2) We define $\partial(r \downarrow \mathscr{R}^-)$ to be the full subcategory of $r \downarrow \mathscr{R}^-$ consisting of non-identity arrows $r \to s$. The *rth matching object of F* is the limit

$$M_rF = \lim_{\partial(r\downarrow\mathscr{R}^-)} F \in \mathscr{C}.$$

This is functorial in F and there is a natural transformation $(-)_r \to M_r$. An arrow $f : F \to G$ in $\mathscr{C}^{\mathscr{R}}$ is a *Reedy fibration* if each corner map

$$F_r \to M_rF \times_{M_rG} G_r,$$

$r \in \mathrm{ob}\mathscr{R}$, is a fibration in $\mathscr{C}$.

(3) We call a map $f : F \to G$ in $\mathscr{C}^{\mathscr{R}}$ an *objectwise weak equivalence* if $f_r : F_r \to G_r$ is a weak equivalence for all $r \in \mathrm{ob}\mathscr{R}$.

Theorem 5.2 (*[19, Theorems 15.3.4, 15.3.15, 15.6.27]*)**.** *Suppose $\mathscr{C}$ is a model category.*

(1) *The category $\mathscr{C}^{\mathscr{R}}$ of diagrams has a model category structure with cofibrations the Reedy cofibrations, fibrations the Reedy fibrations, and weak equivalences the objectwise weak equivalences.*

(2) *If $\mathscr{C}$ is cofibrantly generated, the Reedy model structure on $\mathscr{C}^{\mathscr{R}}$ is cofibrantly generated as well.*

(3) *In $\mathscr{C}^{\mathscr{R}}$, an arrow $f : F \to G$ is an acyclic cofibration if and only if each corner map $L_rG \coprod_{L_rF} F_r \to G_r$ is an acyclic cofibration for all $r \in \mathrm{ob}\mathscr{R}$. Dually, f is an acyclic fibration if and only if each corner map $F_r \to G_r \times_{M_rF} M_rG$ is an acyclic fibration.*

Recall that if $\mathscr{R}$ is a Reedy category and $X \in \widehat{\mathscr{R}}$, then $\mathscr{R} \downarrow X$ is a Reedy category as well.

Proposition 5.3. *Suppose $\mathscr{R}$ is an* EZ *Reedy category and $X \in \widehat{\mathscr{R}}$. Then if $Z \in \mathrm{ob}\mathscr{C}$ is fibrant, the constant diagram $\mathscr{R} \downarrow X \to \mathscr{C}$ on the object Z is Reedy fibrant.*

Proof. This is a straightforward generalization of [19, Proposition 15.10.4]. Let $c_Z : \mathscr{R} \downarrow X \to \mathscr{C}$ denote the constant diagram on Z. We need to check that for all $f : [r] \to X$, $Z \to M_f(c_Z)$ is a fibration in $\mathscr{C}$. Recall that the fth matching object is computed by a limit indexed on $\mathscr{I} = \partial(f \downarrow (\mathscr{R} \downarrow X)^-)$. If $\mathscr{I}$ is empty, then $M_r(c_Z) = *$ and $Z \to *$ is a fibration by assumption. Suppose $\mathscr{I}$ is nonempty. Using the notation of Section 3, suppose $[r] \to [s_i] \to X$ are two arrows in $\mathscr{I}$. We may take the absolute pushout of $r \to s_1$ and $r \to s_2$ in $\mathscr{R}$:

$$\begin{tikzcd}
{[r]} \arrow[r, "\sigma_1"] \arrow[d, "\sigma_2"'] & {[s_1]} \arrow[d, "\tau_2"] \arrow[ddr, bend left] & \\
{[s_2]} \arrow[r, "\tau_1"] \arrow[drr, bend right] & {[t]} \arrow[dr, dotted, "\exists!"] & \\
& & X.
\end{tikzcd}$$

Note that τ_1 and τ_2 are in $\mathscr{R}^-$. Hence $\mathrm{N}\mathscr{I}$ is connected. (In fact, $\mathscr{I}$ has a terminal object given by the EZ decomposition of f.) Write $\pi : \mathscr{I} \to *$; the functor π is thus left cofinal, so $\mathrm{id} \to \pi_*\pi^*$ is a natural isomorphism [26]. Hence $Z \to M_r(c_Z)$ is isomorphic to the identity map on Z, so it is a fibration. $\square$

Let $\partial(\mathscr{R} \downarrow r)$ denote the category of $\mathscr{R}$-simplices $\mathscr{R} \downarrow \partial[r]$. The category $\partial(\mathscr{R} \downarrow r)$ is the full subcategory of $\mathscr{R} \downarrow r$ whose objects are the arrows $x \to r$ for which there exists a factorization $x \to s \to r$ with $\deg s < \deg r$.

Lemma 5.4 (*[19, Proposition 15.2.8]*). *Suppose $\mathscr{R}$ is a Reedy category and $r \in \mathrm{ob}\mathscr{R}$. The inclusion functor*

$$j : \partial(\mathscr{R}^+ \downarrow r) \to \partial(\mathscr{R} \downarrow r)$$

is homotopy right cofinal.

Proof. For (1), let $f : x \to r$ be a non-identity map in $\mathscr{R}$. We factor f as $f = f^+f^-$, $f^+ \in \mathscr{R}^+$, $f^- \in \mathscr{R}^-$. Suppose

$$\begin{tikzcd}
x \arrow[rr, "f"] \arrow[dr, "k"'] & & r \\
& s \arrow[ur, "\ell^+ \in \mathrm{ar}\mathscr{R}^+"'] &
\end{tikzcd}$$

is an object in $f \downarrow j$. We factor $k = k^+k^-$, so $f = (\ell^+k^+)k^-$. Since $\mathscr{R}$ is Reedy, $k^- = f^-$ and $\ell^+k^+ = f^+$, so the triangle

$$\begin{tikzcd}
x \arrow[rr, "f"] \arrow[dr, "f^-"'] & & r \\
& s' \arrow[ur, "f^+"'] &
\end{tikzcd}$$

is terminal in $f \downarrow j$. Hence $\mathrm{N}(f \downarrow j)$ is contractible. $\square$

Proposition 5.5. *Suppose $\mathscr{R}$ is an* EZ *Reedy category and $X \in \widehat{\mathscr{R}}$. Let $\widetilde{X}$ be the diagram*

$$\mathscr{R} \downarrow X \xrightarrow{\pi} \mathscr{R} \xrightarrow{[-]} \widehat{\mathscr{R}}$$

For an r-simplex $f : [r] \to X$, the Reedy map $L_f\widetilde{X} \to \widetilde{X}_f$ is isomorphic to the inclusion $\partial[r] \to [r]$.

Proof. The forgetful functor

$$u : \partial\big((\mathscr{R} \downarrow X) \downarrow f\big) \to \partial(\mathscr{R} \downarrow r)$$

has a left adjoint sending $j : s \to r$ to

$$[s] \xrightarrow{[j]} [r] \xrightarrow{f} X,$$

so u is (homotopy) right cofinal and the map

$$\operatorname*{colim}_{\partial\left((\mathscr{R}\downarrow X)\downarrow f\right)} \widetilde{X} \to \operatorname*{colim}_{\partial(\mathscr{R}\downarrow r)}[-]$$

is an isomorphism. By Lemma 5.4, the Reedy map $L_f\widetilde{X} \to \widetilde{X}_f$ is thus isomorphic to the map

$$\int^{s\in\mathrm{ob}\mathscr{R}} \big(\partial[r]\big)(s) \times [s] \to \int^{s\in\mathrm{ob}\mathscr{R}} [r](s) \times [s].$$

This is precisely the map $\partial[r] \to [r]$. □

Corollary 5.6. *Suppose $X \in q\mathbf{Set}$. Then the natural map*

$$\operatorname*{hocolim}_{\square^n\to X} \square^n \to X$$

is an ∞-equivalence.

Proof. Recall that $\mathscr{Q}$ is EZ and Reedy. By Proposition 5.3, the adjoint pair

$$\mathrm{colim} : q\mathbf{Set}^{\mathscr{Q}\downarrow X} \rightleftarrows q\mathbf{Set} : c$$

is a Quillen adjunction, so we may use the Reedy model structure on $q\mathbf{Set}^{\mathscr{Q}\downarrow X}$ to compute homotopy colimits. The canonical diagram taking $\square^n \to X$ to $\square^n$ is Reedy cofibrant by Proposition 5.5. Hence

$$\operatorname*{hocolim}_{\square^n\to X} \square^n \to \operatorname*{colim}_{\square^n\to X} \square^n \cong X$$

is an ∞-equivalence. □

Corollary 5.6, found in [9] records one of the most important properties of $q\mathbf{Set}$: every cubical set is the homotopy colimit of its cubes. Using Cisinski's terminology, the spatial model structure on $q\mathbf{Set}$ is *regular*. As we will see below, $q_\Sigma\mathbf{Set}$ is regular as well, but this is significantly more difficult to prove.

5.1. Regularity in $q_\Sigma\mathbf{Set}$

In the remainder of this section, we will show that

$$\operatorname*{hocolim}_{\square^n_\Sigma\to X} \square^n_\Sigma \to X$$

is an ∞-equivalence for all extended cubical sets X, i.e., that all extended cubical sets are regular. Our proof uses the *internal nerve* construction of Cisinski [9] and Jardine [23]:

Definition 5.7. Suppose $\mathscr{I}$ is a small category and $\mathscr{C}$ is a cofibrantly generated model category. The *internal nerve* of $\mathscr{I}$ in $\mathscr{C}$ at an object X is the homotopy colimit $\mathrm{hocolim}_{\mathscr{I}} X$ of the constant diagram at X. We denote this by $\mathrm{N}_{\mathscr{C},X}\mathscr{I}$. Writing p for the projection $\mathscr{I} \to *$, we have $\mathrm{N}_{\mathscr{C},X} = \mathbf{L}p_!p^*X$. When X is the terminal object $*$, we will abbreviate $\mathrm{N}_{\mathscr{C}}\mathscr{I} = \mathrm{N}_{\mathscr{C},*}\mathscr{I}$.

Example 5.8. In $s\mathbf{Set}$, $\mathrm{N}_{s\mathbf{Set}}\mathscr{I}$ is weakly equivalent to the nerve of $\mathscr{I}$. Using the simplicial replacement model for the homotopy colimit, these are isomorphic.

Remark 5.9. Internal nerve, as we have defined it, is not functorial. What we have is the following: suppose $f : \mathscr{A} \to \mathscr{B}$ is a functor between small categories. The triangle

$$\begin{tikzcd} \mathscr{A} \arrow[rr, "f"] \arrow[dr, "p"'] & & \mathscr{B} \arrow[dl, "q"] \\ & * & \end{tikzcd}$$

yields a natural transformation $\mathbf{L}p_!p^* \to \mathbf{L}q_!q^*$ since $q_!f_! \cong p_!$ and $p^* = f^*q^*$. This may be used to give $\mathrm{N}_{\mathscr{C},X}$ the structure of a suitably weak 2-functor. We will not need that here; we will write $\mathrm{N}_{\mathscr{C}}f : \mathrm{N}_{\mathscr{C}}\mathscr{A} \to \mathrm{N}_{\mathscr{C}}\mathscr{B}$ below, but we will be careful not to compose maps.

Proposition 5.10. *Suppose $f : \mathscr{A} \to \mathscr{B}$ is a functor between small categories. Then $\mathrm{N}_{q\mathbf{Set}}f$ and $\mathrm{N}_{q_\Sigma\mathbf{Set}}f$ are ∞-equivalences if and only if f is a Thomason equivalence.*

Proof. In $q\mathbf{Set}$, $|-|$ coincides with its left derived functors as everything is cofibrant. This may not be the case in $q_\Sigma\mathbf{Set}$. However, $|-|^{\mathbf{L}}_{q_\Sigma\mathbf{Set}}$ preserves and reflects weak equivalences, where $|-|^{\mathbf{L}}_{q_\Sigma\mathbf{Set}}$ denotes the left derived functor of extended realization. We have squares

$$\begin{tikzcd} \mathrm{N}_{s\mathbf{Set}}\mathscr{A} \arrow[r, "\mathrm{N}_{s\mathbf{Set}}f"] \arrow[d, "\sim"'] & \mathrm{N}_{s\mathbf{Set}}\mathscr{B} \arrow[d, "\sim"] \\ |\mathrm{N}_{q\mathbf{Set}}\mathscr{A}| \arrow[r, "|\mathrm{N}_{q\mathbf{Set}}f|"'] & |\mathrm{N}_{q\mathbf{Set}}\mathscr{B}| \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathrm{N}_{s\mathbf{Set}}\mathscr{A} \arrow[r, "\mathrm{N}_{s\mathbf{Set}}f"] \arrow[d, "\sim"'] & \mathrm{N}_{s\mathbf{Set}}\mathscr{B} \arrow[d, "\sim"] \\ |\mathrm{N}_{q_\Sigma\mathbf{Set}}\mathscr{A}|^{\mathbf{L}}_{\Sigma} \arrow[r, "|\mathrm{N}_{q_\Sigma\mathbf{Set}}f|^{\mathbf{L}}_{\Sigma}"'] & |\mathrm{N}_{q_\Sigma\mathbf{Set}}\mathscr{B}|^{\mathbf{L}}_{\Sigma} \end{tikzcd}$$

commuting up to natural weak equivalence, so $\mathrm{N}_{q\mathbf{Set}}f$ and $\mathrm{N}_{q_\Sigma\mathbf{Set}}f$ are ∞-equivalences if and only if f is a Thomason equivalence. $\square$

Remark 5.11. Proposition 5.10 is part of a general yoga of categorical homotopy theory due to Cisinski [9] and Jardine [23]: the homotopy theory of categories (i.e., spaces) intervenes in every model category via the internal nerve.

Proposition 5.12. *Suppose X is an extended cubical set. The natural map*

$$\operatorname*{hocolim}_{\square^n_\Sigma \to X} \square^n_\Sigma \to X$$

is a natural ∞-equivalence.

Proof. By Corollary 5.6, the map

$$\operatorname*{hocolim}_{\square^n \to i^*X} \square^n \to i^*X$$

is an ∞-equivalence in $q\mathbf{Set}$. Since $i_!$ is left Quillen and all cubical sets are cofibrant, the map

$$\operatorname*{hocolim}_{\square^n \to i^*X \in \mathscr{Q}\downarrow i^*X} \square^n_\Sigma \to i_!i^*X$$

is an ∞-equivalence in $q_\Sigma\mathbf{Set}$. Let G denote the canonical diagram of cubes of X:

$$\mathscr{Q}_\Sigma \downarrow X \xrightarrow{\pi} \mathscr{Q}_\Sigma \xrightarrow{r} q_\Sigma\mathbf{Set}.$$

Recall that i induces a functor $j : \mathscr{Q} \downarrow i^*X \to \mathscr{Q}_\Sigma \downarrow X$ and that j is a Thomason equivalence by Proposition 4.10. Note that $F = Gj$ is roughly the diagram of cubes of i^*X: it is the functor

$$\mathscr{Q} \downarrow i^*X \xrightarrow{\pi} \mathscr{Q} \xrightarrow{r} q\mathbf{Set} \xrightarrow{i_!} q_\Sigma\mathbf{Set}.$$

The natural transformation $\mathbf{L}j_!j^* \to \mathrm{id}$ induces the left arrow in

$$\begin{tikzcd} \operatorname*{hocolim}_{\mathscr{Q}\downarrow i^*X} F \arrow[r] \arrow[d] & \operatorname*{colim}_{\mathscr{Q}\downarrow i^*X} F \arrow[r, equal] \arrow[d] & i_!i^*X \arrow[d, "\varepsilon_X"] \\ \operatorname*{hocolim}_{\mathscr{Q}_\Sigma\downarrow X} G \arrow[r] & \operatorname*{colim}_{\mathscr{Q}_\Sigma\downarrow X} G \arrow[r, equal] & X, \end{tikzcd}$$

which commutes up to natural ∞-equivalence. Thus it is sufficient to show that

$$\operatorname*{hocolim}_{\mathscr{Q}\downarrow i^*X} F \to \operatorname*{hocolim}_{\mathscr{Q}_\Sigma\downarrow X} G$$

is an ∞-equivalence. Let $*$ denote the constant diagram on the terminal object in $q_\Sigma\mathbf{Set}$; then

$$\begin{tikzcd} \operatorname*{hocolim}_{\mathscr{Q}\downarrow i^*X} F \arrow[r] \arrow[d] & \operatorname*{hocolim}_{\mathscr{Q}\downarrow i^*X} * \arrow[d] \\ \operatorname*{hocolim}_{\mathscr{Q}_\Sigma\downarrow X} G \arrow[r] & \operatorname*{hocolim}_{\mathscr{Q}_\Sigma\downarrow X} * \end{tikzcd}$$

commutes up to natural ∞-equivalence. The horizontal arrows are ∞-equivalences since hocolim is a homotopy functor and $\square^n_\Sigma \to *$ is an ∞-equivalence. The right vertical arrow is $\mathrm{N}_{q_\Sigma\mathbf{Set}}j$; this is an ∞-equivalence by Proposition 5.10. $\square$

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Appendix. Day convolution

We briefly review some enriched category theory and introduce some notation. Suppose $\mathscr{V}$ is a closed symmetric monoidal category with all small limits and colimits and $\mathscr{I}$ a small $\mathscr{V}$-category [25]. Write $[-] : \mathscr{I} \to \widehat{\mathscr{I}}$ for the enriched Yoneda embedding, $\widehat{\mathscr{I}} = \mathscr{V}^{\mathscr{I}^{\mathrm{op}}}$. Recall that $[-]$ displays $\widehat{\mathscr{I}}$ as the "free cocompletion" of $\mathscr{I}$: the category of indexed colimit-preserving functors out of $\widehat{\mathscr{I}}$ is equivalent to the category of functors out of $\mathscr{I}$ [25,26]. Here, "indexed colimit" is the usual notion in enriched category theory [25, Chapter 3]. In particular, a $\mathscr{V}$-functor mapping out of a $\mathscr{V}$-cocomplete category preserves indexed colimits if and only if it preserves $\mathscr{V}$-tensor products and ordinary (enriched) conical colimits.

Suppose $(\mathscr{I}, \otimes, e)$ is monoidal. By the universal property of the Yoneda embedding we mentioned above, there is a monoidal structure on $\widehat{\mathscr{I}}$, unique up to unique isomorphism, with the following properties:

(1) $[i] \otimes [j] = [i \otimes j]$ for $i, j \in \mathscr{I}$.
(2) $- \otimes -$ is $\mathscr{V}$-cocontinuous (i.e., preserves all indexed colimits) in each variable.

The canonical presentation of a presheaf as a colimit of representables gives the coend formula

$$(X \otimes Y)_k \cong \int^{i,j\in\mathscr{C}} \mathscr{I}(k, i \otimes j) \otimes (X_i \otimes Y_j). \tag{A.1}$$

The unit is the representable presheaf $[e]$. For fixed $X \in \widehat{\mathscr{I}}$, the functors $X \otimes -$ and $- \otimes X$ both have right adjoints. If $\mathscr{I}$ is symmetric, then the product on $\widehat{\mathscr{I}}$ is closed symmetric monoidal. The hom functor $[-, -]$ is given by the end

$$[X, Y]_i = \int_{\ell\in I} \mathscr{V}(X_\ell, Y_{i\otimes\ell}). \tag{A.2}$$

The product $\otimes$ on $\widehat{\mathscr{I}}$ is known as *Day convolution;* it was introduced in Day's thesis [10]. Im and Kelly prove the following result in [21]; it is an application of the Yoneda lemma.

Proposition A.1. *Suppose $(\mathscr{C}, \otimes, e)$ is a monoidal $\mathscr{V}$-category with small indexed colimits so that $- \otimes -$ preserves indexed colimits in each variable. Given a strong monoidal functor $F : \mathscr{I} \to \mathscr{C}$, the unique cocontinuous extension $\widehat{F} : \widehat{\mathscr{I}} \to \mathscr{C}$ is strong monoidal. If $\mathscr{C}$ and $\mathscr{I}$ are symmetric monoidal categories and F is symmetric strong monoidal, then $\widehat{F}$ is symmetric strong monoidal as well.*

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